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Licensing Strategies in the Presence of Patent Thickets

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Many key industries (e.g., biomedical, pharmaceuticals, telecommunications, and information technologies) are characterized by cumulative innovations, where the introduction of a new product or service often requires many complementary technologies. When these technologies are protected by intellectual property rights owned by many firms, patent thickets exist, which researchers have argued may hinder the development of cumulative innovations. Specifically, patent thickets may lead to excessive royalty burdens for potential licensees, which is called "royalty stacking," and if such costs are passed on to consumers, prices of products based on cumulative technologies will be driven up, dubbed as "double marginalization." The literature, however, does not address these issues under different forms of licensing contracts.

This article develops a game-theoretic model where a downstream firm seeks to license N patents that read on its product from upstream firms. It discusses a variety of licensing forms widely used in practice and attempts to discover whether royalty stacking and double marginalization occur under these forms of licenses. It also studies the impact of bargaining power between parties. It is found that when patent ownership becomes more fragmented, neither royalty stacking nor double marginalization occurs under profit-based royalty, fixed fee, and hybrid licenses. Such problems occur only under pure quantity-based or pure revenue-based royalty licenses when the downstream firm's bargaining power is low. It is also shown that no matter how fragmented the ownership structure of patent is, hybrid licenses consisting of a fixed fee and a quantity- or revenue-based royalty rate lead to the same market outcomes as a fully integrated firm that owns all the patents and the downstream market.

This article has interesting implications for both research and practice. First, the results show that even under the same patent ownership structure, different forms of licenses lead to quite different market outcomes. Therefore, it is suggested that firms and policy makers pay more attention to contractual forms of licenses when trying to minimize the negative impact of patent thickets. Second, the extant literature has largely assumed that quantity-based royalties are used, where double marginalization is the most severe. In practice, revenue-based royalties are most common, under which double marginalization is much milder. Third, the results show that patent pools can be most effective in mitigating royalty stacking and double marginalization when quantity-based or revenue-based royalties are the sole or primary payment form, especially when downstream firms have low bargaining power.

Many key industries (e.g., biomedical, pharmaceuticals, telecommunications, and information technologies) are characterized by cumulative innovations, where the introduction of a new product or service often requires many complementary technologies. To develop new products and services, firms often need to access innovations owned by many other firms (Bessen and Maskin, 2009; Grindley and Teece, 1997). For example, more than 7,000 patents have been declared as essential to the third generation (3G) cellular technologies, and among these, 1,259 representative patents are owned by 41 companies (Goodman and Myers, 2005). Software development is another extreme case of cumulative innovation. Since computer software became patentable in the United States in the 1990s, software patents granted increased dramatically in the last decade, and firms developing new software often have to license patents essential to their development efforts from multiple entities (Bessen and Hunt, 2003; Hall and MacGarvie, 2006).

On the one hand, firms with key patents can reap significant value by licensing their intellectual property (IP) rights to other firms developing complementary technologies. For example, Qualcomm owns several hundreds of patents for the Code Division Multiple Access (CDMA) wireless technologies and has licensed its essential patent portfolio to more than 100 telecommunications equipment manufacturers worldwide. Qualcomm earned more than one-third of its total revenues from licensing of its technologies (Associated Press Financial Wire, July 25, 2007). For

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another example, IBM is a pioneer in the patenting and licensing of software. It owns about 40,000 patents, more than any other company in the world, and earns approximately $2 billion of licensing revenues a year (Preston, 2005; Stone, 2004). Patent revenues may be vital even for smaller companies. For example, in 2006, Apple Computers paid Creative Technology, a Singapore-based company holding patents in digital music player technologies, a $100 million licensing fee, which made up 90% of Creative's profit for the year and kept the struggling firm afloat (Burns, 2007; Noguchi, 2006).

On the other hand, patents already granted can make the development of cumulative technologies prohibitively expensive, especially when the innovator needs to license patents held by many firms or faces potential litigation from patent holders. Related, and often overlapping, patents owned by many entities are often described as "patent thickets" and researchers have argued that patent thickets can be detrimental to innovation, especially in information industries such as software (see, among others, Heller and Eisenberg, 1998; Lessig, 2001; Shapiro, 2001; Bessen and Maskin, 2009). One notable, and perhaps unexpected, example of high licensing costs for downstream firms is Microsoft. As one of the largest software companies, Microsoft needs to license many patented software components from independent software vendors (ISV). In 2005, Microsoft paid about $1 billion to license intellectual property from other companies while collecting only $100 million in royalties on its own patents (Ricadela, 2006).

For firms either trying to gain access to patented technologies or with patents to license, a key issue is to negotiate an optimal licensing contract. Central to most negotiations are the form and the terms of the payment: the parties must agree on the form of payment, such as an upfront fee, running royalties, a share of profits, or any combination of the above, as well as the amount or rate of these payments. The first key set of research questions this article attempts to answer is: When one firm licenses many patented technologies, what is the optimal form and term of a license? Do the optimal strategies change when patents are held by many firms rather than very few firms?

Another important set of issues this article addresses is: Does the existence of patent thickets deter innovation? Specifically, do patent thickets lead to less production of innovative products, given that firms choose their licensing strategies optimally? Heller and Eisenberg (1998) argue that innovations protected by excessive property rights (such as patents) tend to be under-used, a phenomenon they aptly call the "tragedy of the anti-commons." Fragmented ownership of patents may cause the *royalty stacking* problem: the more firms own patents, the higher the licensing costs for downstream firms (Lemley and Shapiro, 2007). As Heller and Eisenberg (1998) put it, "Each upstream patent allows its owner to set up another tollbooth on the road to product development, adding to the cost and slowing the pace of downstream . . . innovation." Another problem is *double marginalization*: Shapiro (2001) shows that when a downstream product needs access to multiple patents, the more firms holding these patents, the higher the price of the downstream product. (The double marginalization problem was originally studied by Cournot [1838] in the context of complementary goods.) Royalty stacking and double marginalization result in low quantities of the products that utilize upstream innovations, a.k.a. the tragedy of the anti-commons. It must be noted that Shapiro (2001) studies only one form of license, namely a quantity-based royalty license, where the licensee pays a royalty for each unit of its output. It is then natural to ask: Do royalty stacking and double marginalization occur under other forms of licenses? More importantly, do these problems occur under optimal license agreements?

This paper models a variety of licensing forms under different market structures in the context of patent thickets. In the model, N essential patents read on a downstream product. First presented is the case where all N patents are owned by one upstream firm and thus can be licensed in one agreement; then examined is the case where N upstream firms each own a patent and a separate license is needed for each patent. It is found that licensing strategies can be more important than the market structure (a.k.a., the number of firms/licenses) in shaping market outcomes. The impact of bargaining power on firms' licensing strategies and market outcomes is also discussed.

BIOGRAPHICAL SKETCH

Dr. Lihui Lin is assistant professor of information systems at the School of Management, Boston University. She received her Ph.D. in economics from the University of Texas at Austin. Her research interests concern economic issues in technological innovation and knowledge transfer, focusing on contractual design and the impacts of uncertainty, network effects, and asymmetric information. Her work has appeared in journals such as *Management Science, MIS Quarterly, Journal of Management Information Systems, Managerial Finance, Electronic Commerce Research and Applications*, and *International Journal of Management and Network Economics*.

Some interesting results are obtained, which contribute to the literature in several ways. First, this article introduces some widely used licensing forms that have not been studied before in the literature, and shows that the choice of licensing form is pivotal for firms' decisions and market outcomes. It must be pointed out that the royalty licenses studied in the literature are in fact quantity-based, which is only one of several types of royalty licenses (see Kamien [1992], and Kamien and Tauman [2002] for surveys of the literature). In practice, sales revenues are often used as bases for calculating royalties. Battersby and Grimes (2005) observe that a revenue-based royalty rate is "by far the most common form of compensation charged by licensors" (pp. 2–3). Port et al. (2005, p. 150) also report the same fact. Brunsvold and O'Reilley (2004) illustrate how to clearly define a royalty base in a license agreement (pp. 115–16) and explain why a royalty based on the sales revenues of products can be particularly useful in licensing research tools (pp. 116–18). For example, Qualcomm charges licensees a worldwide standard royalty rate that is less than 5 percent of the wholesale selling price of a licensed handset (Qualcomm, 2008). In IP litigation, profits are often used as the royalty base when the court decides on the compensation for the patent owner (Goldscheider, 1995; Goldscheider, Jarosz, and Mulhem, 2005; Slind-Flor, 2004). Quantity-based, revenue-based, and profit-based royalties differ in at least two important ways: (1) a quantity-based royalty rate is *a monetary amount* the licensee pays per unit of its output, while a revenue- or profit-based royalty rate is a *percentage* of the licensee's sales revenues or profits; (2) these royalty bases offer quite different incentives to licensees, leading to very different market outcomes, as later shown in this article. In practice, for a particular product or service, one type of royalty may be easier to calculate than others. In addition to royalty licenses, there are many other forms of licenses in practice and firms often combine different forms of compensation (Battersby and Grimes, 2005; Port et al., 2005). License agreements may also serve purposes other than mere access to protected technologies. For example, Kulatilaka and Lin (2006) show how complex license agreements (upfront payments plus a royalty cap) can be used by an innovator to preempt potential competitors, as well as to obtain funds for development efforts. Licensing of patents can also be part of (sometimes even a reason for) mergers and acquisitions. This paper studies licensing forms commonly used in practice, and shows that market outcomes vary significantly with the form of license.

Second, the results show that royalty stacking or double marginalization does *not* occur for a variety of licensing forms. In particular, when fixed-fee licenses, profit-based royalty licenses, or hybrid licenses combining a fixed fee and a royalty (quantity- or revenue-based) are used, fragmented ownership of patents in the upstream market does not lead to royalty stacking or double marginalization in the downstream market. For any of these licensing forms, no matter how many firms hold patents essential to the downstream product, the price and quantity of the product remain the same. In other words, whether the patents are licensed separately or as a pool, the outcome in the downstream market is unchanged. Moreover, under hybrid licenses the price and quantity in the downstream market are the same as those in the case of a fully integrated firm that owns both the N upstream patents and the downstream market (i.e., no license is needed). And under fixed-fee or profit-based royalty licenses, the price in the downstream market is even lower than that charged by a fully integrated firm. Note that these results apply to any distribution of bargaining power between upstream and downstream firms. In sum, fragmented patent ownership may cause royalty stacking and double marginalization only under pure quantity-based or revenue-based royalty licenses, but it happens only when the downstream firm's bargaining power is low.

Third, the difference in marginal costs of upstream firms may lead to different optimal licensing strategies. This suggests that firms in different industries should adopt licensing strategies that are best suited for their cost structures. For example, software companies are known to have marginal costs close to zero, and the semiconductor industry also tends to have low marginal cost, while industries such as chemical and manufacturing may have higher marginal costs. Therefore, information technology companies should not simply follow the common practice in traditional industries when drafting and negotiating license agreements.

Model Setup

Suppose one firm (call it the downstream firm) needs to license N patents in order to develop and commercialize a new product (also referred to as the "final product"). The firm has monopoly rights in the product market; for example, it can also patent this final product. It incurs a marginal cost of a for each unit produced. The demand for this product exhibits constant price elasticity, $q \equiv D(p) = Ap^{-\varepsilon}$, where p is the price and ε is the absolute value of the price elasticity ($\varepsilon > 1$ by the assumption that the firm is a monopolist). This functional form is used because the discussion focuses on pricing rules, and holding the elasticity constant allows us to compare the pricing rules, which are expressed in terms of elasticity. Simulations show that the key results in this paper are also true qualitatively for other functional forms of the demand function. Although the downstream firm is a monopolist, it may face competition from firms offering imperfect substitutes. For example, Apple's iPod products have unique designs and functionalities, yet there are other MP3 players that compete with them. The competitiveness of the market is captured by price elasticity, where a higher elasticity (absolute value) indicates more competition.

In general, to access the N patents that read on the final product, the downstream firm needs M license agreements, where one agreement may cover one or more patents ($M \leq N$). First, the case where one license agreement covers all N patents ($M = 1$) is studied, then the general case of $1 < M \leq N$ is discussed.

The downstream firm may either license the rights to use the patented technologies or purchase components with patents embedded. For example, in semiconductor, telecommunication, chemical, and biomedical industries, firms owning patents often provide downstream firms with components such as integrated chips or chemical agents that embed patented technologies. In the software industry and e-business, however, the patented technologies are usually in digital form and thus the marginal cost of, say, providing a software component, is essentially zero. Assume that for each unit of the final product produced, it costs the upstream firm that owns the k-th patented technology $c^k \geq 0$ ($k = 1, \ldots, N$) to provide it. The overall marginal costs in the upstream for each unit of the final product is $\sum_{k=1}^{N} c^k$.

For succinctness, define the ratio between the marginal cost of providing the k-th patent and the downstream marginal cost as $m^k \equiv c^k/a$, and the ratio between the total upstream marginal costs and the downstream marginal cost $m \equiv \sum_{k=1}^{N} c^k/a \equiv \sum_{k=1}^{N} m^k$.

Now consider the bargaining power between firms. The bargaining power of the upstream firms tends to be high. A patent owner does not have to license its technology to a particular firm unless there is only one potential licensee (even in that case, there can be new entrants to the market willing to license from the patent holder). And unless it grants exclusive rights to one party, a patent holder can license to any number of firms. Furthermore, when a firm holds key patents to a technology, it is often difficult for others to invent around, and if others are found infringing on the firm's patents, the firm is entitled to compensation. Even if no infringing is found, litigation itself (or even a threat of litigation) can benefit the patent-holding firm, and most patent disputes are settled outside the court, with a downstream firm paying a large amount to the patent holder. A case in point is the patent dispute between Research In Motion (RIM), the provider of BlackBerry wireless email services, and NTP, a patent-holding company. After years of litigation, in March 2006, RIM settled the dispute for \$612.5 million; under the agreement, even if the U.S. Patent and Trademark Office eventually overturned NTP's patents, NTP would not have to repay the \$612.5 million.

The bargaining power of the downstream firm depends on the alternatives it has. To introduce its product, the downstream firm needs the N patented technologies; it can either: (1) license from the patent-holding firms; (2) license from other firms providing substitutable technologies; or (3) attempt to develop the technologies itself.

When the downstream firm chooses to develop the technologies in-house, the costs can be prohibitively high, and the R&D efforts may fail. Even if the R&D is successful and an innovation is achieved, it may be infringing on extant patents. If, however, the firm does successfully "invent around" the patented technologies and is not infringing on any previous patents, the downstream firm can expect to earn the profit of a "fully integrated" firm (which owns all the upstream patents and produces the downstream product) net of the investment, which can serve as a reservation payoff for the downstream firm. Another possibility is that there are other firms providing similar technologies (which is not often the case), and the downstream firm will be able to choose licensors that offer better terms. Thus, offers made by other upstream firms providing competing technologies can also shape the downstream firm's reservation payoff.

Therefore, the downstream firm's reservation payoff is a measure of its bargaining power. Denote the downstream firm's reservation payoff by Π^o, which can be considered as $\Pi^o = \max\{0, T^{II} - \mathit{Investment},$ expected profits when licensing from other firms}, where T^{II} represents the profits of a fully integrated firm.

Our model assumes that patent-holding firms make take-it-or-leave-it (TIOLI) offers, and the downstream firm agrees to license if it expects to earn at least its reservation payoff. Note that this assumption can be considered as capturing the last phase of real negotiations. In practice, negotiations over licenses between firms may take many rounds, where firms make offers and counter-offers. If the offers are unacceptable to the downstream firm, it can threaten to leave, and hence the negotiation will continue until the upstream firms make offers that are acceptable to the downstream firm. Through these negotiations, the downstream firm's reservation payoff can be revealed, and in the end patent-holding firms will make offers that provide the downstream firm with its reservation payoff and hence the downstream firm will accept.

In the next two sections, the licensing strategies and market outcomes are examined, for any reservation payoff the downstream firm may have, including the case of $\Pi^o = 0$ (i.e., the upstream firms have full bargaining power, which may happen when the investment needed to develop the N technologies exceeds T^{II} and there are no substitutable technologies available).

One License Agreement ($M = 1$) Covering N Patents

This section discusses the case of $M = 1$, where one license agreement covers all N patents. Typically, this happens when the N patents are owned by one firm, referred to as the *integrated* upstream firm. The results derived here also apply to cases where multiple patent-owning firms act as one entity and license N patents as a pool.

The sequence of events goes as follows: In the first stage, the integrated upstream firm and the downstream firm negotiate a license agreement covering the N patents. In the second stage, the downstream firm produces the final product, choosing a price that maximizes its profits. Under each form of license, the firms' optimal decisions are derived; in other words, the Nash equilibrium is shown. The last subsection compares the market outcomes under these forms of license and discusses firms' optimal choices of licensing forms and terms.

A benchmark case is presented in Appendix 1A, where a fully (both horizontally and vertically) integrated firm owns all the patents and produces the final product and thus no license is needed. The price and profit of a fully integrated firm are derived (see Appendix 1A), which are compared with the cases below.

Quantity-Based Royalty Licenses

Now suppose the integrated upstream firm licenses its N patents at a single royalty rate of u per unit of the downstream output. In this paper, "per-unit royalty" and "quantity-based royalty" are used interchangeably. The model is solved by backward induction. At the second stage, the downstream firm chooses its price to maximize its profit, taking the royalty rate as given; at the first stage, the integrated upstream firm chooses a royalty rate that maximizes its own profits, subject to the downstream firm's participation (incentive compatibility) constraint. The solutions are derived in Appendix 1B (the results are shown in Table 1).

Downstream firm with no bargaining power. First, consider the case where the downstream firm has no bargaining power. The results show that the upstream firm's optimal royalty rate allows *the downstream firm with no bargaining power to earn a strictly positive payoff*. In other words, the upstream firm could have

Table 1. Market Outcome under One Quantity-Based Royalty License Covering N Patents

	$0 \leq \Pi^o \leq \Pi^{IU}_{up}$ (*)	$\Pi^{IU}_{up} < \Pi^o \leq T^{II}$ (**)
Contract term u^{IU}	$u^{IU} = \frac{1}{\varepsilon-1}(1+\varepsilon m)a \equiv u^{IU}_{up}$	$u^{IU} = (1-1/\varepsilon)\left(\frac{A}{\Pi^o \varepsilon}\right)^{1/(\varepsilon-1)} - a$
Price p^{IU}	$p^{IU} = \left(\frac{\varepsilon}{\varepsilon-1}\right)^2 (1+m)a \equiv p^{IU}_{up}$	$p^{IU} = \left(\frac{A}{\Pi^o \varepsilon}\right)^{1/(\varepsilon-1)}$
Upstream profit π^{IU}	$\pi^{IU} = \frac{A(\varepsilon-1)^{2\varepsilon-1}}{[a(1+m)]^{\varepsilon-1}\varepsilon^{2\varepsilon}} \equiv \pi^{IU}_{up}$	$\pi^{IU} = \varepsilon \Pi^o \left[1 - \left(\frac{\varepsilon \Pi^o}{A}\right)^{1/(\varepsilon-1)}(1+m)a\right] - \Pi^o$
Downstream profit Π^{IU}	$\Pi^{IU} = \Pi^{IU}_{up}$	$\Pi^{IU} = \Pi^o$

* $\Pi^{IU}_{up} = \frac{A(\varepsilon-1)^{2\varepsilon-2}}{[a(1+m)]^{\varepsilon-1}\varepsilon^{2\varepsilon-1}}$

** $T^{II} = \frac{A(\varepsilon-1)^{\varepsilon-1}}{[a(1+m)]^{\varepsilon-1}\varepsilon^{\varepsilon}}$

charged a rate higher than the optimal level and the downstream firm would still accept it. The reason for the upstream firm not charging a higher rate is that a per-unit royalty rate has two opposite effects on the licensor's payoff: while a higher rate allows the licensor to collect more licensing revenues per unit, it has a negative effect on the royalty base because it forces the licensee to raise the price of the product, causing the quantity sold to drop. The optimal royalty rate balances these two effects, given the pricing rule of the downstream firm.

Downstream firm with any reservation payoff. Next, consider this problem for any possible reservation payoff of the downstream firm. First, note that even when the downstream firm has a zero reservation payoff (no bargaining power), it earns a positive profit given by Π_{up}^{IU}. Therefore, as long as the downstream firm's reservation payoff is lower than Π_{up}^{IU}, the upstream firm keeps charging its optimal royalty rate, leaving the downstream firm with the same payoff of Π_{up}^{IU}. Note that it is in the best interest of the upstream firm to leave the downstream firm with a payoff higher than its bargaining power. Thus the market outcome, including the price of the final product and the profits of the two firms, remains the same as the case where the downstream firm has no bargaining power.

When the downstream firm has a reservation payoff higher than Π_{up}^{IU}, in other words, it can bargain a higher payoff than what the upstream firm is willing to let it have; the downstream firm's profit will equal its reservation payoff. In this regime, not surprisingly, the royalty rate decreases as the downstream firm's bargaining power increases. As a result, the higher the downstream firm's bargaining power, the lower the price of the final product. This implies that the consumers of the final product will benefit when the provider has more bargaining power, because it can negotiate a lower royalty rate, reducing its costs and price (see Figure 1).

Figure 1 plots the price of the final product under the different licensing forms against the reservation payoff of the downstream firm, Π^o (see Appendix 2 for proof), where panel (a) shows the case of $m>0$ and (b) $m=0$. In both panels of Figure 1, the price under quantity-based royalty license clearly shows two regimes: it remains constant when the downstream firm's bargaining power (Π^o) is low, while it decreases when the downstream firm's bargaining power is high.

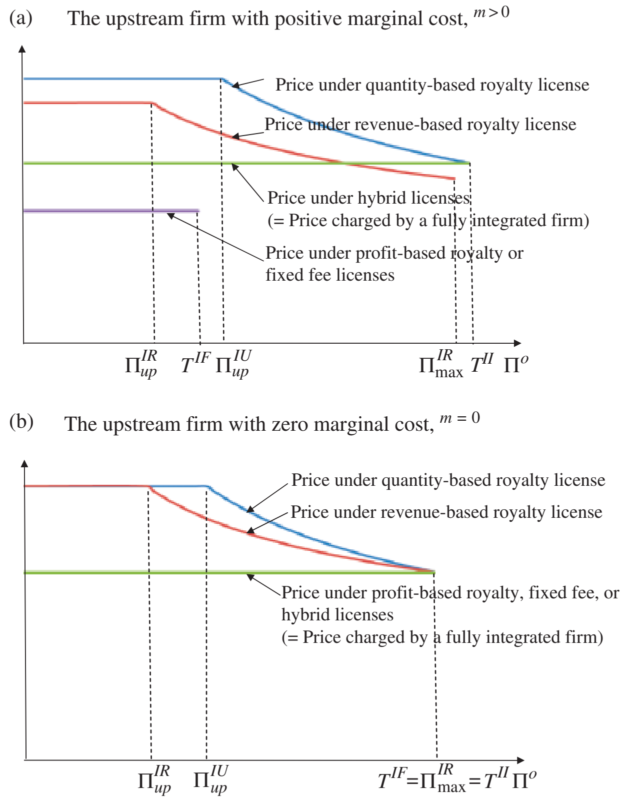

Figure 1. One Downstream Firm and One Integrated Upstream Firm Owning N Patents: The Price of the Final Product under Different Forms of License

Figure 2 shows the profit of the integrated upstream firm under different forms of license. Again, the profit for the upstream firm under a quantity-based license has two regimes: it remains constant when Π^o is low while it decreases with Π^o when Π^o is high. Note that in the regime where the upstream firm's profit remains constant, higher bargaining power does not always lead to higher payoff for the upstream firm, while lower bargaining power does not necessarily mean

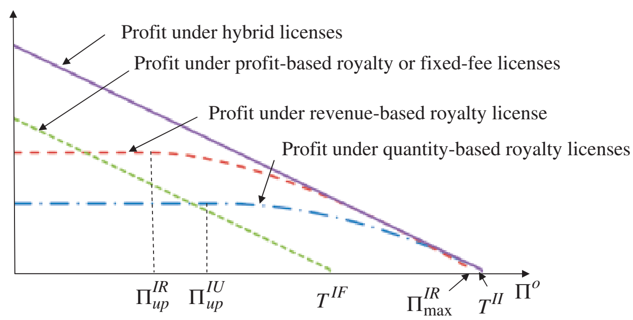

Figure 2. The Profit of the Integrated Upstream Firm under Different Forms of License

lower payoff for the downstream firm. It is often taken for granted that a firm with bargaining power extracts all possible surpluses from the other firm. However, the results show that this is not the case with a quantity-based royalty license.

Revenue-Based Royalty Licenses

Suppose the upstream firm charges a revenue-based royalty rate: for each dollar of revenue the final product generates, the downstream firm pays the upstream firm r fraction. Again, the model is solved by backward induction. Just as in the case of quantity-based royalty license, at the second stage, the downstream chooses its profit-maximizing price, taking the royalty rate r as given; at the first stage, the integrated upstream firm chooses its profit-maximizing royalty rate, subject to the downstream firm's participation constraint. However, a revenue-based royalty influences the profit functions of the two firms differently, providing different incentives to the firms than in the case of a quantity-based royalty. See Appendix 1C for the detailed solution method and the results (Table 2).

Similar to the case of a per-unit royalty license, the outcomes show two regimes. When Π^o is low ($0 \leq \Pi^o < \Pi_{up}^{IR}$), the downstream firm's participation constraint is not binding; the optimal royalty rate, and the optimal price, and the two firms' profits remain the same in this regime. In particular, even when it has *no bargaining power* ($\Pi^o = 0$), the downstream firm still earns a positive payoff of Π_{up}^{IR}. Again this is because the upstream firm faces a trade-off when determining the royalty rate. A higher revenue-based royalty rate increases the licensor's per-dollar payoff for each dollar of downstream sales revenues, but may decrease the royalty base (here revenues) due to a higher price of the final product. Therefore, an integrated upstream firm does not charge too high a rate even when it has the bargaining power to do so; and the upstream firm's optimal rate leaves the downstream firm with a positive payoff.

In the regime where the downstream firm's bargaining power is sufficiently high, the royalty rate, the price, and the profit of the upstream firm are all decreasing functions of the downstream firm's bargaining power.

In Figure 1, the price under revenue-based royalty license clearly demonstrates these two regimes, and so is the profit for the upstream firm under a revenue-based license in Figure 2.

It should be noted that the results under quantity-based and revenue-based royalties also differ in the following ways: (1) the cutoff points of the two regimes are different, with $\Pi_{up}^{IR} < \Pi_{up}^{IU}$; and (2) the upper limits on the downstream firm's reservation payoff (which is also the maximum payoff the downstream firm may get) differ, with $\Pi_{\max}^{IR} \leq T^{II}$.

Profit-Based Royalty Licenses

Suppose the license specifies that the upstream firm gets g fraction of the downstream firm's profits. (The symbol g and later the superscript G are used for the lack of other suitable letters, and after Goldscheider, an advocate of profit-based royalties.) Interestingly, with a profit-based royalty license, the price of the final product is independent of the royalty rate and the distribution of bargaining power. The price depends solely on the downstream firm's marginal cost and the elasticity in the downstream market. The reason is that when the royalty is tied to profits rather than quantity or revenues, the licensee's profit-maximizing pricing decisions are not distorted by the license at all. With a quantity-based or revenue-based

Table 2. Market Outcome under One Revenue-Based Royalty License Covering N Patents

	$0 \leq \Pi^o \leq \Pi_{up}^{IR}$ (*)	$\Pi_{up}^{IR} < \Pi^o \leq \Pi_{\max}^{IR}$ (**)
Contract term r^{IR}	$r^{IR} = \frac{m(\varepsilon-1)+1}{m(\varepsilon-1)+\varepsilon} \equiv r_{up}^{IR}$	$r^{IR} = 1 - \varepsilon\left(\frac{\Pi^o}{A}\right)^{1/\varepsilon}\left(\frac{a}{\varepsilon-1}\right)^{1-1/\varepsilon}$
Price p^{IR}	$p^{IR} = \frac{\varepsilon(\varepsilon m-m+\varepsilon)}{(\varepsilon-1)^2}a \equiv p_{up}^{IR}$	$p^{IR} = \left[\frac{Aa}{\Pi^o(\varepsilon-1)}\right]^{1/\varepsilon}$
Upstream profit π^{IR}	$\pi^{IR} = \frac{A(\varepsilon-1)^{2\varepsilon-2}}{a^{\varepsilon-1}\varepsilon^{\varepsilon}[\varepsilon+m(\varepsilon-1)]^{\varepsilon-1}} \equiv \pi_{up}^{IR}$	$\pi^{IR} = \Pi^o\left[\left(\frac{A}{\Pi^o}\right)^{1/\varepsilon}\left(\frac{\varepsilon-1}{a}\right)^{1-1/\varepsilon} - \varepsilon - m\varepsilon + m\right]$
Downstream profit Π^{IR}	$\Pi^{IR} = \Pi_{up}^{IR}$	$\Pi^{IR} = \Pi^o$

$*: \Pi_{up}^{IR} = \frac{A(\varepsilon-1)^{2\varepsilon-1}}{a^{\varepsilon-1}\varepsilon^{\varepsilon}[\varepsilon+m(\varepsilon-1)]^{\varepsilon}}$

$**: \Pi_{\max}^{IR} \equiv \frac{A(\varepsilon-1)^{\varepsilon-1}}{a^{\varepsilon-1}(\varepsilon+m\varepsilon-m)^{\varepsilon}}$

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Table 3. Market Outcomes under One Profit-Based Royalty License and One Fixed-Fee License Covering N Patents*

	Profit-based royalty license	Fixed-fee license
Contract term	$g^{IG} = 1 - \frac{a^{\varepsilon-1}\varepsilon^{\varepsilon}\Pi^{o}}{A(\varepsilon-1)^{\varepsilon-1}}$	$F^{IF} = \frac{A(\varepsilon-1)^{\varepsilon-1}}{a^{\varepsilon-1}\varepsilon^{\varepsilon}} - \Pi^{o}$
Price	$p^{IG} = \frac{\varepsilon}{\varepsilon-1}a$	$p^{IF} = \frac{\varepsilon}{\varepsilon-1}a$
Upstream profit	$\pi^{IG} = T^{IG} - \Pi^{o}$	$\pi^{IF} = T^{IF} - \Pi^{o}$
Downstream profit	$\Pi^{IG} = \Pi^{o}$	$\Pi^{IF} = \Pi^{o}$

**Note:* These results are valid for $0 \le \Pi^{o} \le T^{IG} = T^{IF}$, where $T^{IG} = T^{IF} = \frac{A(\varepsilon-1)^{\varepsilon-1}}{a^{\varepsilon-1}\varepsilon^{\varepsilon}}[1 - m(\varepsilon - 1)]$.

royalty license, however, the licensee's profit maximization condition is distorted by the royalty rate (a quantity-based royalty raises the marginal cost of the downstream firm while a revenue-based royalty lowers the marginal revenue), which is related to the upstream firm's marginal costs and the bargaining power.

For any given reservation payoff for the downstream firm, Π^{o}, the outcome is derived and shown in Appendix 1D (Table 3). Obviously, under a profit-based royalty license, it is optimal for the upstream firm to charge the highest possible rate, extracting all possible surpluses from the downstream firm, which means that the downstream firm always gets its reservation payoff. Note that the negotiations of the royalty rate do not change the total profits of the two firms, given by T^{IG}, which means one firm's gain is the other firm's loss. The royalty rate simply determines how the two firms split the total profits.

It must be noted, however, that unlike the other two types of royalty license discussed earlier, a profit-based royalty license may not be feasible for certain parameter values.

Lemma 1: *Profit-based royalty licenses are feasible only when the marginal costs in the upstream firms are relatively low:* $\sum_{k=1}^{N} c^{k} < \frac{1}{\varepsilon-1}a$, *or equivalently,* $m < \frac{1}{\varepsilon-1}$.

Proof: *Firms won't consider using a profit-based license unless there is a positive profit to split, so it must be the case that* $T^{IG} > 0$, *which means that* $m(\varepsilon - 1) < 1$. *Q.E.D.*

Lemma 1 shows that a profit-based royalty license is feasible only when the upstream costs are relatively low. The downstream firm charges a price that does not take the upstream costs into account and thus is quite low (note that $p^{IG} \le p^{II}$), so the profit share the upstream firm gets may not cover its costs. Therefore, the upstream costs need to be low enough for a profit-based royalty to yield a positive payoff for the upstream firm.

In Figure 1(a), the price of the final product under a profit-based royalty license exists only when such a license is feasible. Specifically, it is feasible when the downstream firm's reservation payoff does not exceed the total profits of the two firms. In Figure 1(b), since the upstream costs are zero, a profit-based royalty license is always feasible, and thus the price has value for all possible values of the downstream firm's reservation payoff.

The profit of the integrated upstream firm under a profit-based royalty license shown in Figure 2 is a downward-sloping straight line. This is simply because the total profit of the two is constant and the two firms are splitting it based on their bargaining power (recall that the horizontal axis is the downstream firm's reservation payoff). Note that given the same bargaining power of the downstream firm, the upstream firm's profit under different licenses ranks differently. This again shows that the form of license provides different incentives to firms and leads to different outcomes.

Fixed-Fee Licenses

Suppose the license takes the form of a fixed amount F paid by the downstream firm to the upstream firm. The results show that a fixed-fee license leads to exactly the same price as a profit-based license does ($p^{IF} = p^{IG}$). With a fixed-fee license, just like a profit-based royalty license, the price of the downstream product is not distorted by upstream costs.

The market outcome under a fixed-fee license for any Π^{o} is obtained (see Appendix 1D for the results). Similar to the case of a profit-based royalty license, the upstream firm charges the highest possible licensing fee, and the downstream firm earns exactly its reservation payoff. Also, the total profits of firms are the same as that in the case of a profit-based royalty license ($T^{IF} = \pi^{IF} + \Pi^{IF} = T^{IG}$), and remain constant for any level of the fixed fee.

Lemma 2: *Fixed-fee licenses are feasible only when the marginal costs in the upstream firms are relatively low:* $\sum_{k=1}^{N} c^{k} < \frac{1}{\varepsilon-1}a$, *or equivalently,* $m < \frac{1}{\varepsilon-1}$.

The results also show that holding the downstream firm's reservation payoff constant, the upstream firm is indifferent between a fixed-fee license and a profit-based royalty license.

Hybrid Licenses: Royalty plus Fixed Fee

Now, consider licenses that combine royalty rates with fixed fees. Note that a hybrid license combining a profit-based royalty with a fixed fee is not discussed, because for a given distribution of bargaining power between the firms, any combination of these two elements will lead to the same downstream price and the same payoffs to the firms. So, hybrid licenses refer to licenses consisting of a fixed-fee and a quantity-based or revenue-based royalty rate.

First, suppose the license specifies a per-unit royalty rate u and a fixed fee F. The downstream firm's optimal pricing decision depends on the per-unit royalty part only, and thus the decision rule is the same as that in a pure quantity-based royalty license. The upstream firm maximizes its profit by choosing a per-unit royalty rate u and a fixed fee F, subject to the downstream firm's participation constraint.

Next, consider a hybrid license consisting of a royalty rate r for each dollar of sales revenue and a fixed fee F. Understandably, the downstream firm's optimal pricing rule is the same as that in a pure revenue-based royalty license. The upstream firm's problem is similar to the above case.

The solutions to the above two problems are provided in Appendix 1E (in Table 4, H in the superscript indicates a "hybrid" license, U a per-unit royalty rate, and R a revenue-based royalty rate). Clearly, the two types of hybrid licenses lead to the same price and the same profits for the firms. Moreover, the price equals that charged by a fully integrated firm ($p^{IHU} = p^{IHR} = p^{II}$), and the sum of the two firms' profits equals the profit of a fully integrated firm. This means that two firms optimally using a hybrid license lead to the same market outcome as a fully integrated firm.

Figure 1 shows the price of the final product under a hybrid license. In Figure 2, the profit of the integrated upstream firm under hybrid licenses is again a linear function of the downstream firm's reservation payoff shown. This is because the two firms are splitting the total profit, which equals that of a fully integrated firm, based on their bargaining power.

It should be noted that the fixed fee part in these two types of licenses is different, with the fee in the revenue-based royalty hybrid license lower ($F^{IHR} < F^{IHU}$). In a hybrid license with a per-unit royalty rate, it is optimal for the upstream firm to use royalties to cover the marginal costs only (note that $u^{IHU} = \sum_{k=1}^{N} c^k$) and rely on the fixed fee to get a positive payoff, while in a hybrid license with a revenue-based royalty rate, the optimal royalties exceed the costs, allowing the upstream firm to charge a lower fixed fee.

Market Outcomes under Different Licensing Forms and Optimal Strategies

First, compare the price of the final product under different licensing forms. Figure 1 shows that for any valid Π^o, a quantity-based royalty license leads to the highest price. With a quantity-based royalty rate, the higher the output, the more the downstream firm pays in royalties, thus the firm has the strongest incentive to raise price and limit quantity. A revenue-based royalty leads to a lower price than a quantity-based royalty does ($p^{IR} < p^{IU}$). When the downstream firm increases its output, the price will be lower due to a downward-sloping demand, thus the revenues never increase as much as the quantity, and so neither does the revenue-based royalty burden. Therefore, the

Table 4. Market Outcomes under One Hybrid License Covering N Patents*

	Fixed fee plus a quantity-based royalty	Fixed fee plus a revenue-based royalty
Contract terms	$u^{IHU} = ma \equiv \sum_{k=1}^{N} c^k,\ F^{IHU} = T^{II} - \Pi^o$	$r^{IHR} = \frac{m}{1+m},\ F^{IHR} = \frac{T^{II}}{1+m} - \Pi^o$
Price	$p^{IHU} = \frac{\varepsilon}{\varepsilon-1}(1+m)a$	$p^{IHR} = \frac{\varepsilon}{\varepsilon-1}(1+m)a$
Upstream profit*	$\pi^{IHU} = T^{II} - \Pi^o$	$\pi^{IHR} = T^{II} - \Pi^o$
Downstream profit	$\Pi^{IHU} = \Pi^o$	$\Pi^{IHR} = \Pi^o$

* *Note:* These results are valid for $0 \le \Pi^o \le T^{II}$, where $T^{II} = \frac{A(\varepsilon-1)^{\varepsilon-1}}{[a(1+m)]^{\varepsilon-1}\varepsilon^{\varepsilon}}$.

downstream firm has more incentive to increase production and reduce price under a revenue-based license.

The price under hybrid licenses is always lower than that under a quantity-based license, and most of the time lower than that under a revenue-based license. This is because hybrid licenses have a fixed-fee part, and thus do not rely solely on the royalty to reap values from the license, and therefore the royalty rate is lower than in pure-form licenses. Since the downstream firm's pricing rule is only distorted by the quantity- or revenue-based royalty rate, a lower royalty rate usually implies less distortion, and a lower price.

It has been shown that the two types of hybrid licenses result in the same price as the one charged by a fully integrated firm. Note that a fully integrated firm does not always lead to the lowest price. In particular, a profit-based royalty or a fixed-fee license leads to a lower price than that charged by a fully integrated firm. This is because under a profit-based royalty or a fixed-fee license the downstream firm's pricing rule only considers its own marginal cost and excludes the upstream costs, while a fully integrated firm takes into account the marginal costs both upstream and downstream. This result implies that the patented technologies are in fact used more when licensed under a profit-based royalty or a fixed-fee license than when owned in-house.

Also, the equilibrium price under a profit-based royalty, a fixed-fee, or a hybrid license remains unchanged for different distributions of bargaining power between firms. For quantity- or revenue-based royalty licenses, the price decreases with the downstream firm's bargaining power when the downstream firm's bargaining power is high enough.

Next, compare the integrated upstream firm's profits under different licensing forms (see Figure 2). The following proposition provides the optimal licensing strategies for an integrated upstream firm that makes a TIOLI offer to the downstream firm with a reservation payoff.

Proposition 1: *For any reservation payoff of the downstream firm,* Π^o*, an integrated upstream firm's optimal licensing form is a hybrid license that consists of a fixed fee and a quantity- or revenue-based royalty rate. The optimal licenses are:*

(1) $u^{IHU} = ma = \sum_{k=1}^{N} c^k$ *and* $F^{IHU} = \frac{A(\varepsilon-1)^{\varepsilon-1}}{[a(1+m)]^{\varepsilon-1}\varepsilon^{\varepsilon}} - \Pi^o$*; or*

(2) $r^{IHR} = \frac{m}{1+m}$ *and* $F^{IHR} = \frac{A(\varepsilon-1)^{\varepsilon-1}}{a^{\varepsilon-1}(1+m)^{\varepsilon}\varepsilon^{\varepsilon}} - \Pi^o$. (See Appendix 3A for proof.)

This result shows that hybrid licenses are optimal. Recall that the price under hybrid licenses is higher than profit-based royalty and fixed-fee licenses, but mostly lower than quantity- and revenue-based royalty licenses. The royalty burden under quantity- and revenue-based royalties is too high, causing the downstream firm to charge too high a price. Under profit-based royalty and fixed-fee licenses, however, the downstream firm ignores the upstream costs and thus charges too low a price.

Under the optimal license, the royalty rate is either ma per unit of output, or $\frac{m}{1+m}$ fraction of the sales revenue. In both cases, the royalty rate increases with the marginal cost ratio between the upstream and the downstream firms, m. Also, the fixed-fee part of the optimal license (F^{IHU} and F^{IHR}) decreases with m. This means that the higher the relative marginal costs in the upstream, the more the upstream firm relies on the royalties to extract licensing revenues.

In the special case where $m = 0$, the two types of optimal hybrid licenses become the same (royalty rates both become 0, and the fixed fees are equal), and are equivalent to the optimal pure fixed-fee license. Since the optimal pure fixed-fee license yields the same results as the optimal pure profit-based royalty license, the following corollary is derived.

Corollary 1: *When* $m = 0$*, the optimal licensing strategy for an integrated upstream firm is a fixed-fee license with a fee of* $F^{IF} = \frac{A(\varepsilon-1)^{\varepsilon-1}}{a^{\varepsilon-1}\varepsilon^{\varepsilon}} - \Pi^o$ *or a profit-based royalty license with a royalty rate of* $g^{IG} = 1 - \frac{a^{\varepsilon-1}\varepsilon^{\varepsilon}\Pi^o}{A(\varepsilon-1)^{\varepsilon-1}}$. (See Appendix 3B for proof.)

The market outcomes under the optimal hybrid license are the same as that under a fully integrated firm, which leads to the following proposition (the intuition is explained in the discussion of hybrid licenses):

Proposition 2: *When one integrated upstream firm owns N patents and chooses the optimal licensing strategies described in Proposition 1, the equilibrium price of the downstream product is the same as that charged by a fully integrated firm:* $p = \frac{\varepsilon}{\varepsilon-1}(1+m)a$*; and the sum of the profits of the downstream firm and the integrated upstream firm also equals that of a fully integrated firm.* (See Appendix 3C for proof.)

The above results have implications to the understanding of networks effects. The upstream firm is providing complementary technologies to the downstream firm, so the downstream market can be viewed as a network exhibiting indirect network effects. Since the optimal licensing strategy leads to the same price as a fully integrated firm does, the quantity of output in the downstream market is also the same. This means that under the optimal licensing strategy, two firms providing complementary goods can create the same network size as a single, fully integrated firm does.

This suggests that the same market outcome can be achieved under different market structures. From a policy point of view, for industries characterized by cumulative innovations and network effects, the form of contract can be as important as the market structure in shaping the market outcome.

While the optimal licensing strategy is to use a hybrid license, in practice licenses in pure forms are still widely observed, and most often licenses use a revenue-based royalty rate. This may be due to sundry reasons. The downstream firm may simply be unable to pay (or commit to) a fixed fee upfront; it can only pay back the upstream firm once it receives revenues from sales, and thus it makes more sense to specify a royalty rate. Another possible reason is that the fixed-fee component in the optimal license is usually difficult to negotiate. To determine the fixed fee, firms must agree on the size of the market (represented by the parameter A). On the other hand, in pure quantity- or revenue-based royalty licenses, the optimal terms depend only on the price elasticity of the demand in the downstream market (ε) and the relative marginal costs of firms (m) and thus it is easier to reach an agreement.

Extension: Equity Holding between Firms

This subsection discusses briefly the case of partial equity ownership. In practice, especially in biotechnology, purchase of equity is often part of complex licensing deals (e.g., Seget, 2005). Researchers have also studied both licensing and acquisition as commercialization strategies of start-up firms (Gans, Hsu, and Stern, 2002; Levine, 2009).

First, suppose the integrated upstream firm owns e fraction of the equity of the downstream firm. If the equity provides the upstream firm with *no other rights* than receiving e fraction of the downstream profits, then the price in the downstream market will be the same as that under a *profit-based royalty license* ($p = \frac{\varepsilon}{\varepsilon-1}a$; see Appendix 4). This is because the downstream firm chooses its profit-maximizing price without any distortion from the upstream firm. When e is close to 1, however, it is unrealistic to believe that the upstream firm plays no role in determining the downstream price and quantity. When $e = 1$, the upstream firm has essentially acquired the downstream firm and the price will be the same as that charged by a fully integrated firm, $p = \frac{\varepsilon}{\varepsilon-1}(1+m)a$. Therefore, it is likely that as e increases (the upstream firm owning more shares of the downstream firm), the price rises from the level for two firms using a profit-based royalty license (or a fixed-fee license) to that for a fully integrated firm (or two firms using an optimal hybrid license).

Now suppose the downstream firm acquires e fraction of the integrated upstream firm's equity. The downstream firm pays a lump sum of S to the upstream firm for the equity, and gains access to upstream technologies for no further charges. The resulting downstream price is given by $p = \frac{\varepsilon}{\varepsilon-1}(1+e\cdot m)a$ (see Appendix 4). Thus, as e increases from 0 to 1, the price again increases from the level under a profit-based royalty (or a fixed-fee) license to that for a fully integrated firm.

In sum, partial equity ownership by either of the firms leads to an outcome between the case of a profit-based royalty (or a fixed-fee) license and the case of a fully integrated firm (or two firms using an optimal hybrid license), and the higher the equity share, the higher the price in the downstream market.

M ($1 < M \leq N$) License Agreements Covering N Patents

This section discusses the case when the N patents (with a marginal cost of $c^k \geq 0$, $k = 1, \ldots, N$ each) are owned by M ($1 < M \leq N$) upstream firms. Suppose that the i-th upstream firm owns a set of patents denoted by I and the marginal cost of providing these patents is $c_i = \sum_{k\in I} c^k = \sum_{k\in I} m^k a \equiv m_i a$. The downstream firm needs a separate license with each of the M firms. In order to compare the results here with those under one license, assume that all M licenses are of the same form when discussing a specific licensing form.

The sequence of events is: at the first stage, the downstream firm negotiates with M upstream firms *simultaneously*; at the second stage, the downstream firm chooses the optimal price of its product and

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Table 5. Market Outcome under M Quantity-Based Royalty Licenses Covering N Patents

	$0 \le \Pi^o \le \Pi_{up}^U$ (*)	$\Pi_{up}^U < \Pi^o \le T^{II}$ (**)
Contract terms u_i^U	$u_i^U = \left[m_i + \frac{1}{\varepsilon - M}(1+m)\right]a \equiv u_{i,up}^U$	$u_i^U = \left[(1-1/\varepsilon)\left(\frac{A}{\Pi^o \varepsilon}\right)^{1/(\varepsilon-1)} - a(1+m)\right] - m_i a$
Royalty burden $\sum_{i=1}^{M} u_i^U$	$\sum_{i=1}^{M} u_i^U = \frac{1}{\varepsilon - M}(m\varepsilon + M)a$	$\sum_{i=1}^{M} u_i^U = (1-1/\varepsilon)\left(\frac{A}{\Pi^o \varepsilon}\right)^{1/(\varepsilon-1)} - a$
Price p^U	$p^U = \frac{\varepsilon^2}{(\varepsilon-1)(\varepsilon-M)}(1+m)a \equiv p_{up}^U$	$p^U = \left(\frac{A}{\Pi^o \varepsilon}\right)^{1/(\varepsilon-1)}$
Upstream profit π_i^U	$\pi_i^U = \frac{A(\varepsilon-1)^{\varepsilon}(\varepsilon-M)^{\varepsilon-1}}{[a(1+m)]^{\varepsilon-1}\varepsilon^{2\varepsilon}} \equiv \pi_{i,up}^U$	$\pi_i^U = \frac{\varepsilon \Pi^o}{M}\left[1 - \left(\frac{\varepsilon \Pi^o}{A}\right)^{1/(\varepsilon-1)}(1+m)a\right] - \frac{\Pi^o}{M}$
Downstream profit Π^U	$\Pi^U = \Pi_{up}^U$	$\Pi^U = \Pi^o$

*: $\Pi_{up}^U = \frac{A[(\varepsilon-1)(\varepsilon-M)]^{\varepsilon-1}}{[a(1+m)]^{\varepsilon-1}\varepsilon^{2\varepsilon-1}}$,

**: $T^{II} = \frac{A(\varepsilon-1)^{\varepsilon-1}}{[a(1+m)]^{\varepsilon-1}\varepsilon^{\varepsilon}}$

realizes profits. The Nash equilibrium under each form of license is defined.

Quantity-Based Royalty Licenses

Suppose the downstream firm pays the *i*-th upstream firm a royalty of u_i (covering all patents the *i*-th firm owns) for each unit of the final product. At the second stage, the downstream firm finds its profit-maximizing pricing rule, taking the royalty rates as given. At the first stage, each of the N upstream firms finds its optimal royalty rate, taking the royalty rates charged by other upstream firms and the downstream firm's pricing rule as given.

The solution to this problem is shown in Appendix 1F (results in Table 5). Similar to the case of an integrated firm, the equilibrium results have two regimes: the regime where the downstream firm's participation constraint is non-binding and the one where the constraint is binding (but the cutoff point for the two regimes here is different from that with one license). Another interesting result is that even though upstream firms charge different rates based on their marginal costs, in equilibrium each upstream firm earns the same payoff.

Note that when patent ownership is fragmented and M licenses are needed, it is important to know whether royalty stacking and double marginalization occur. First, the sum of the M per-unit royalty rates, $\sum_{i=1}^{M} u_i^U$, is compared with the per-unit royalty rate charged by an integrated firm u^{IU} (see Figure 3a). It is found that $\sum_{i=1}^{M} u_i^U > u^{IU}$ when $0 \le \Pi^o < \Pi_{up}^{IU}$, while $\sum_{i=1}^{M} u_i^U = u^{IU}$ when $\Pi_{up}^{IU} \le \Pi^o \le T^{II}$. Furthermore, the higher M is, the higher the total royalty rate. This means that fragmented patent ownership results in a higher overall per-unit royalty when the downstream firm's reservation payoff is low; however, when the downstream firm's reservation payoff is high, M licenses impose the same royalty burden on the downstream firm as one license does.

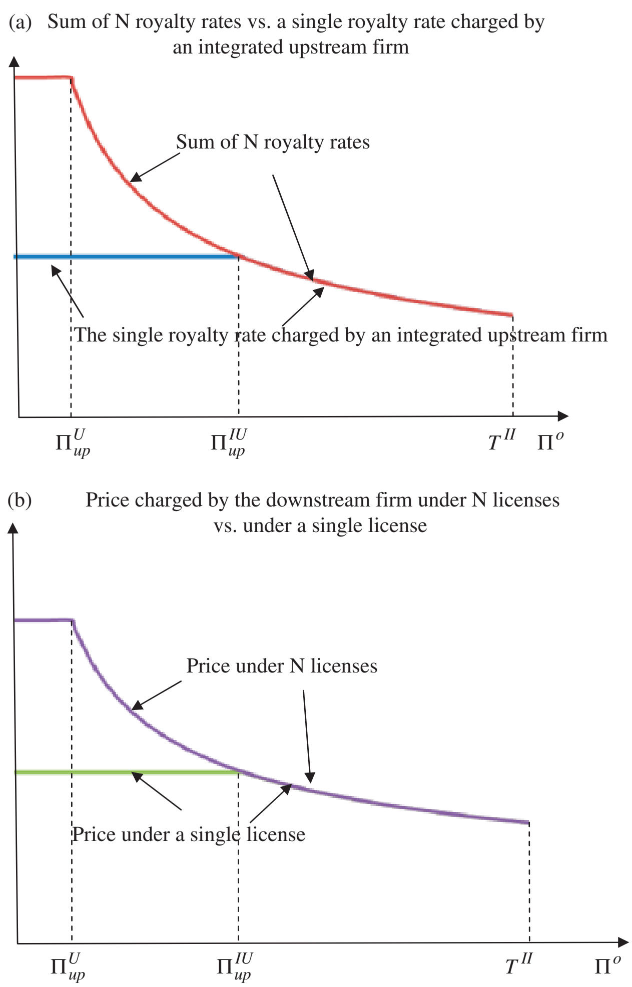

Figure 3. Quantity-Based Royalty Licenses

The intuition is as follows. When the downstream firm has low bargaining power, the royalty rates are determined by the upstream firms' profit maximizing conditions. As found in the literature (e.g., Shapiro, 2001), when multiple upstream firms own N patents, they ignore the complementarity (which is a form of positive network externality; see Economides, 1996) of their patents to other patents, while one firm owning N patents internalizes the complementary effects of the patents. Note that the complementarity (externalities) only exists when each upstream firm has an impact on the quantity decision made by the downstream firm. Therefore, overall the royalty rates charged by M firms are higher than that charged by one firm. However, when the downstream firm has high bargaining power, the royalty rates are no longer determined by upstream firms' own profit maximizing rules. Instead, the rates are determined by the downstream firm's bargaining power, and the downstream firm can negotiate M licenses such that the overall effect of the rates is the same as that under one license.

Multiple licenses lead to a higher price only when the downstream firm's bargaining power is low; however, when the downstream firm has sufficiently high bargaining power, the price under the two scenarios becomes the same. Figure 3b compares the resulting price under N (i.e., M = N) quantity-based royalty licenses with that under one quantity-based license. The reason is simply that when the downstream firm has high bargaining power, the sum of the rates is the same as the rate of one license, which means that the licensing costs are the same, and so is the optimal price.

In sum, quantity-based royalty licenses lead to royalty stacking and double-marginalization problems only when the downstream firm has low bargaining power.

Proposition 3: *When N patents are licensed through M* $(1 < M \leq N)$ *rather than one quantity-based royalty licenses, royalty stacking and double marginalization occur when* $0 \leq \Pi^o < \Pi^{IU}_{up}$, *but do not occur when* $\Pi^{IU}_{up} \leq \Pi^o \leq T^{II}$. (See Appendix 5A for proof.)

Corollary 2: *Under M* $(1 < M \leq N)$ *quantity-based royalty licenses, when royalty stacking and double marginalization do occur, the higher M is, the more severe these problems*. (See Appendix 5A for proof.)

Revenue-Based Royalty Licenses

Suppose each of the M licenses specifies a royalty rate of r_i to be paid to the *i*-th upstream firm for each dollar of revenue the downstream firm earns. Similar to the case of quantity-based royalty licenses, at the second stage the downstream firm finds its price taking the royalty rates as given; at the first stage, each upstream firm finds its optimal royalty rate, taking other upstream firms' rates and the downstream firm's reaction as given. See Appendix 1F (in particular Table 6) for the analytical results. Unlike the case where the M licenses are quantity-based royalties, each upstream firm earns different payoffs when the licenses are revenue-based royalties. Specifically, an upstream firm's profit is proportional to its marginal cost.

Similar to the case of quantity-based royalties, royalty stacking and double marginalization happen

Table 6. Market Outcome under M Revenue-Based Royalty Licenses Covering N Patents

	$0 \leq \Pi^o \leq \Pi^R_{up}$ (*)	$\Pi^R_{up} < \Pi^o \leq \Pi^{IR}_{\max}$ (**)
Contract terms r_i^R	$r^R_{i,up} = \frac{m_i(\varepsilon-1)+1}{(1+m)(\varepsilon-1)+M} \equiv r^R_{i,up}$	$r_i^R = \frac{1}{M} + \varepsilon\left(m_i - \frac{1+m}{M}\right)\left(\frac{\Pi^o}{A}\right)^{1/\varepsilon}\left(\frac{a}{\varepsilon-1}\right)^{1-1/\varepsilon}$
Royalty burden $\sum_{i=1}^{M} r_i^R$	$\sum_{i=1}^{M} r_i^R = \frac{m(\varepsilon-1)+M}{(1+m)(\varepsilon-1)+M}$	$\sum_{i=1}^{M} r_i^R = 1 - \varepsilon\left(\frac{\Pi^o}{A}\right)^{1/\varepsilon}\left(\frac{a}{\varepsilon-1}\right)^{1-1/\varepsilon}$
Price p^R	$p^R = \frac{\varepsilon}{(\varepsilon-1)^2}[(1+m)(\varepsilon-1)+M]a \equiv p^R_{up}$	$p^R = \left[\frac{Aa}{\Pi^o(\varepsilon-1)}\right]^{1/\varepsilon}$
Upstream profit π_i^R	$\pi_i^R = \frac{A[\varepsilon a+(\varepsilon-1)m_i a](\varepsilon-1)^{2\varepsilon-2}}{a^\varepsilon \varepsilon^\varepsilon[(1+m)(\varepsilon-1)+M]^\varepsilon} \equiv \pi^R_{i,up}$	$\pi_i^R = \frac{\Pi^o}{M}\left[\left(\frac{A}{\Pi^o}\right)^{1/\varepsilon}\left(\frac{\varepsilon-1}{a}\right)^{1-1/\varepsilon} - \varepsilon(1+m)\right] + m_i\Pi^o$
Downstream profit Π^R	$\Pi^R = \Pi^R_{up}$	$\Pi^R = \Pi^o$

*: $\Pi^R_{up} = \frac{A(\varepsilon-1)^{2\varepsilon-1}}{a^{\varepsilon-1}\varepsilon^\varepsilon[(1+m)(\varepsilon-1)+M]^\varepsilon}$

**: $\Pi^{IR}_{\max} \equiv \frac{A(\varepsilon-1)^{\varepsilon-1}}{a^{\varepsilon-1}(\varepsilon+m\varepsilon-m)^\varepsilon}$

when the downstream firm's bargaining power is low. Figure 4a shows the sum of the royalty rates charged by N firms (i.e., M = N) and the rate charged by an integrated firm (r^{IR}), all using revenue-based royalty licenses. Figure 4b illustrates the price under these cases.

Proposition 4: *When N patents are licensed through M* ($1 < M \leq N$) *rather than one revenue-based royalty licenses, royalty stacking and double marginalization occur when* $0 \leq \Pi^o < \Pi^{IR}_{up}$*, but do not occur when* $\Pi^{IR}_{up} \leq \Pi^o \leq \Pi^{IR}_{\max}$. (See Appendix 5B for proof.)

Corollary 3: *Under M* ($1 < M \leq N$) *revenue-based royalty licenses, when royalty stacking and double marginalization do occur, the higher M is, the more severe these problems*. (See Appendix 5B for proof.)

The intuition is also similar to the case of quantity-based royalty licenses. Even though the royalty base is different and the firms have different decision rules, it is true for both forms of licenses that the royalty rates are determined purely by the upstream firms' profit-maximizing conditions when the downstream firm has low bargaining power, while the downstream firm's bargaining power determines the rates when it is sufficiently high. Therefore, when M upstream firms own patents, royalty stacking occurs when the downstream firm has low bargaining power but does not when the downstream firm's bargaining power is sufficiently high. The results on price are also similar to the case of quantity-based royalty licenses, for the same reasons.

Because revenue-based royalty licenses are most common in practice while quantity-based royalty licenses are discussed most in the literature, the prices with N licenses (i.e., M = N) under these two forms of licenses are plotted in Figure 5. Figure 5 shows that while both forms lead to double marginalization when the downstream bargaining power is low, the actual price under N revenue-based licenses is much lower than that under N quantity-based licenses. Recall that when a single license covers all N patents, a quantity-based royalty leads to a higher price than a revenue-based royalty does. When multiple licenses are involved and double marginalization occurs, the price difference is magnified.

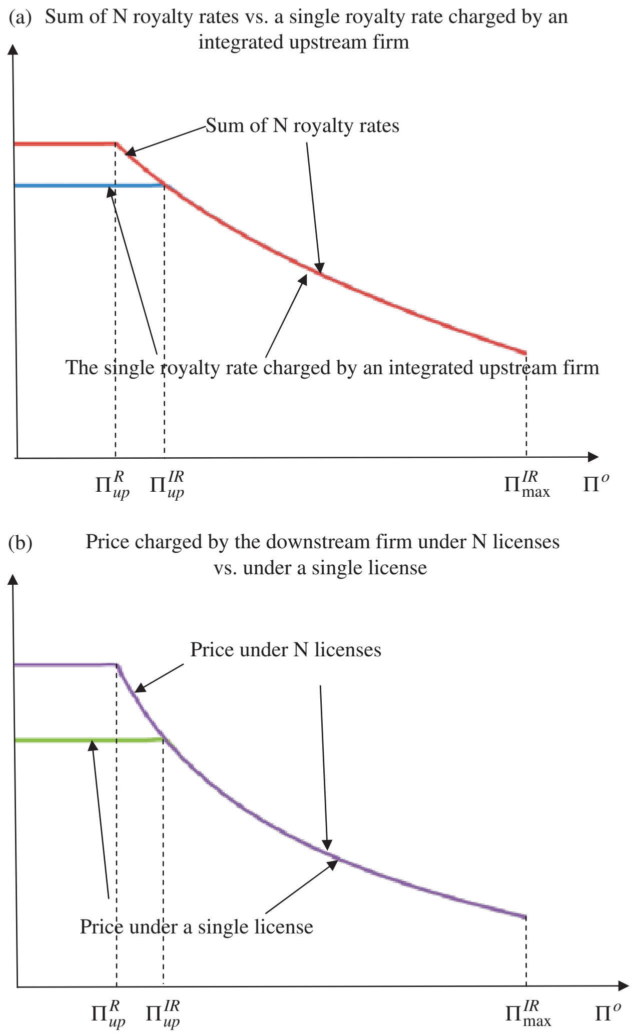

Figure 4. Revenue-Based Royalty Licenses

Profit-Based Royalty Licenses

Suppose each of the M licenses requires that the *i*-th upstream firm gets g_i fraction of the downstream

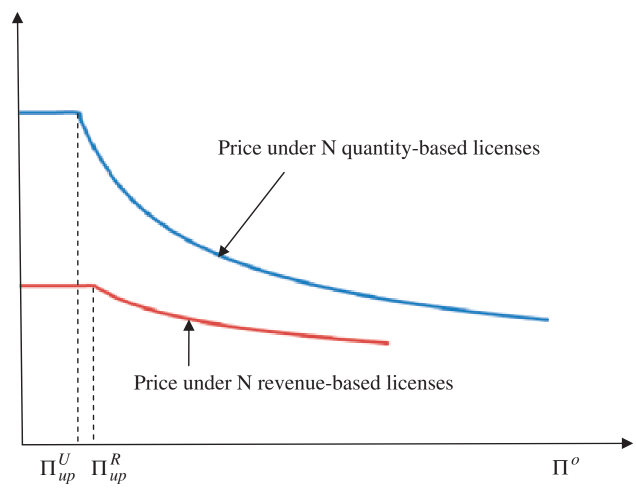

Figure 5. Price under N Quantity-Based Royalty Licenses versus Price under N Revenue-Based Royalty Licenses

firm's profits. It turns out that the downstream price is totally independent of the number of upstream firms and thus double marginalization does not occur. The intuition for this result is that when the royalty is based on profits, the licensee's profit-maximizing pricing decisions are not distorted by the licenses at all. Therefore, no matter how many upstream firms own patents, the downstream firm charges the same price. In other words, double marginalization does not occur.

The results also show that the profit shares of the downstream firm and the N upstream firms depend on the bargaining power of the downstream firm. The downstream firm always earns exactly its reservation payoff. Therefore, for the downstream firm with a given bargaining power, the overall royalty rate charged by N upstream firms is no different from that charged by one integrated firm.

Proposition 5: *Under profit-based royalty licenses, neither royalty stacking nor double marginalization occurs when multiple licenses are used.*

It must be noted, though, that with M firms holding patents, the downstream firm has to negotiate M licenses rather than one, and thus the transaction costs can be much higher, and may not be negligible.

Fixed-Fee Licenses

When fixed-fee licenses are used, again the licensee's profit-maximizing pricing decisions are not distorted by the licenses, and thus the downstream firm charges the same price for any number of upstream firms owning patents. Therefore, double marginalization does not occur under fixed-fee licenses. The reason is that a fixed-fee license does not impose a marginal cost on the licensee (downstream firm). Double marginalization occurs when upstream firms do not take into account their positive externalities to each other; however, the externalities only exist when each upstream firm has an impact on the quantity decision made by the downstream firm. When fixed-fee licenses are used, since the licensee's marginal cost is not changed by the term of the license, its price and quantity decisions are therefore independent of the licensing terms, and thus the upstream firms no longer impose externalities on each other. The same logic also applies to the case of a profit-based royalty license.

The licensing fees set by the M upstream firms will add up to make the downstream firm's participation constraint binding, just as in the case of one fixed-fee license. In other words, the licensing fees will not stack up.

Proposition 6: *Under fixed-fee licenses, licensing fees do not stack up and double marginalization does not occur when multiple licenses are used.*

Hybrid Licenses: Royalty plus Fixed Fee

Consider licenses that combine royalty rates with fixed fees. Appendix 1F (in particular Table 7) presents the results. For each type of hybrid license, the sum of the royalty rates in M licenses for the N patents equals the single rate in one license, and so does the sum of the fixed fees. In other words, royalty stacking does not happen for any type of hybrid licenses.

Under M hybrid licenses the price remains the same as that under one hybrid license, which is the same as charged by a fully integrated firm (no licensing needed). Therefore, there is no double marginalization.

Proposition 7: *Under hybrid licenses with a fixed fee and a quantity- or revenue-based royalty rate, neither royalty stacking nor double marginalization occurs when multiple licenses are used.*

Market Outcomes and Social Welfare

The market outcomes regarding royalties and prices under different forms of licenses are summarized in Figure 6.

When M upstream firms hold the N patents that read on the final product, the profit of each firm in equilibrium is derived when pure quantity- or revenue-based royalty licenses are used. For all other forms of licenses considered, however, while the total profits of firms can be derived, the profit each upstream firm gets depends on the negotiations and the bargaining power of each upstream firm. Therefore, the optimal licensing strategy for an individual upstream firm is not discussed, but the following proposition and its corollary describe the effects of licenses on the total profits.

Proposition 8: *The optimal hybrid licenses yield the highest total profits of the upstream and downstream*

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Table 7. Market Outcomes under M Hybrid Licenses Covering N Patents

	Fixed fee plus a quantity-based royalty	Fixed fee plus a revenue-based royalty
Contract terms*	$u_i^{HU} = m_i a \equiv c_i$ $F_i^{HU} = T^{II} - \Pi^o - \sum_{j=1, j\neq i}^{M} F_j^{HU}$	$r_i^{HR} = \frac{m_i}{1+m}$ $F_i^{HR} = \frac{T^{II}}{1+m} - \Pi^o - \sum_{j=1, j\neq i}^{M} F_j^{HR}$
Total burden on the downstream firm	$\sum_{i=1}^{M} u_i^{HU} = ma = \sum_{k=1}^{N} c^k$ $\sum_{i=1}^{N} F_i^{HU} = T^{II} - \Pi^o$	$\sum_{i=1}^{M} r_i^{HR} = \frac{m}{1+m}$ $\sum_{i=1}^{M} F_i^{HR} = \frac{T^{II}}{1+m} - \Pi^o$
Price	$p^{HU} = \frac{\varepsilon}{\varepsilon-1}(1+m)a$	$p^{HR} = \frac{\varepsilon}{\varepsilon-1}(1+m)a$
Upstream profit*	$\pi_i^{HU} = F_i^{HU}$ $\sum_{i=1}^{M} \pi_i^{HU} = T^{II} - \Pi^o$	$\pi_i^{HR} = \frac{m_i}{1+m} T^{II} + F_i^{HR}$ $\sum_{i=1}^{M} \pi_i^{HR} = T^{II} - \Pi^o$
Downstream profit	$\Pi^{HU} = \Pi^o$	$\Pi^{HR} = \Pi^o$

$* : T^{II} = \frac{A(\varepsilon-1)^{\varepsilon-1}}{[a(1+m)]^{\varepsilon-1}\varepsilon^{\varepsilon}}$

	Royalty stacking (i.e., sum of M royalty rates is greater than the royalty rate under a single license)	Double marginalization (i.e., price of the final product under M licenses is greater than that under a single license)	Price of the final product compared with other forms of licenses (all under M licenses)	Implications
Quantity-based royalty licenses	Occurs when the downstream firm has low bargaining power	Occurs when the downstream firm has low bargaining power	Highest	Patent pooling would greatly reduce the royalty burden on the licensee, and the price of products using the patents
Revenue-based royalty licenses	Occurs when the downstream firm has low bargaining power	Occurs when the downstream firm has low bargaining power, but less severely than under quantity-based royalty licenses	Lower than under quantity-based royalty licenses	Patent pooling would reduce the royalty burden on the licensee, and the price of products using the patents
Hybrid licenses with a fixed fee and a quantity-(revenue-) based royalty	No royalty stacking, sum of N royalty rates equals the royalty rate under a single hybrid license covering N patents	No double marginalization, price is the same as that under a single hybrid license covering N patents	Lower than under revenue-based royalty licenses	Neutral to patent pooling
Profit-based royalty licenses (equivalent to fixed-fee licenses)	No royalty stacking, sum of N royalty rates equals the royalty rate under a single profit-based royalty license covering N patents	No double marginalization, price is the same as that under a single profit-based royalty license covering N patents	Lower than under hybrid licenses	Neutral to patent pooling

Figure 6. Market Outcomes when M ($1 < M \leq N$) Licenses Are Used to Cover N Patents

firms, which are equal to the total profit of a fully integrated firm. (See Appendix 6A for proof.)

Corollary 4: *By switching from M quantity-based royalty licenses (or M revenue-based royalty licenses) to M hybrid licenses, each firm may achieve the same or higher profit.* (See Appendix 6B for proof.)

Corollary 4 shows that it can be in the firms' best interest to switch from pure quantity-based or revenue-based royalty licenses to hybrid licenses. Such switching also results in a lower price, leading to higher consumer surplus. However, constructing and switching to hybrid licenses require coordination between upstream firms, and thus may not be easily achieved.

In fact, the lack of coordination between upstream firms may be one of the reasons for the wide use of pure quantity-based or revenue-based royalty licenses. One appeal of these two forms of licenses is that each upstream firm sets its rate independent of other negotiations; for both forms, the rate depends on the upstream firm's own marginal cost, the ratio between the sum of upstream marginal costs and downstream cost, and the downstream price elasticity. In all other forms of licenses considered, the terms in one license depend on how much rent other upstream firms can extract from the downstream firm. Therefore, even though hybrid licenses lead to higher total profits of upstream firms, firms may still adopt pure quantity-based or revenue-based royalty licenses, resulting in a case of "prisoners' dilemma."

Concluding Remarks

This study shows that using different forms of license agreements can lead to very different market outcomes. One key result is that patent thickets do not necessarily lead to royalty stacking and double marginalization. There has been debate over the merits of the current patent system, and fragmented ownership of intellectual property rights is often considered to have stifled innovation. The results here, however, show that under certain contractual forms of licenses, market outcomes remain the same for different patent ownership structures, and yet under the same ownership structure, different forms of licenses lead to quite different market outcomes. Therefore, it is suggested that firms and policy makers pay more attention to contractual forms of licenses when trying to minimize the negative impact of patent thickets.

Despite the fact that for many licensing forms fragmented ownership of patents does not lead to royalty stacking or double marginalization, royalty stacking and double marginalization do occur for the most common form of license, namely revenue-based royalty licenses. In the literature, however, royalties are often assumed to be based on quantities, for which double marginalization is the most severe. This article shows that the price under revenue-based royalty licenses tends to be much lower than that under quantity-based royalty licenses, and the price difference is even more significant when multiple licenses are needed. Therefore, even though patent thickets may cause higher prices in products based on cumulative innovations, the resulting prices may be lower than predicted by the extant literature, which is largely based on the assumption of quantity-based royalty licenses.

The results presented in this article can be used to analyze industries where new products often involve the licensing of patented technologies. For example, the patents essentials to the MPEG Surround standard are owned by Dolby Laboratories, Inc., France Telecom, Fraunhofer IIS, LG Electronics, LSI Corporation, and Philips Electronics, and are licensed using quantity-based royalty schedules. Licensees of the MPEG standard, typically less powerful than these patent owners, are developers of end-user products that incorporate the MPEG Surround technologies (Via Licensing Corp, 2008). The results in this study suggest that when patent ownership is fragmented, quantity-based royalty licenses may lead to royalty stacking for licensees and the prices of products reading on these patents may be very high, especially when the downstream firms have low bargaining power relative to the patent owners. In such cases, patent pools can be effective in keeping the royalty burden and the prices low. As it turns out, the owners of the MPEG patents pooled their patents together and licensed them jointly, charging a single quantity-based royalty to licensees. This article suggests that this effort may have avoided royalty stacking and helped keep the prices of end-user products low.

One must also be careful in applying the results in this article to business cases of licensing. Hybrid licenses are often observed in practice. The results in this article show that optimal hybrid licenses do not lead to royalty stacking or double marginalization. However, it must be noted that when a license consists of both a fixed fee and a royalty, it does not necessarily mean that the *optimal* hybrid license is used. When the fixed fees are much lower than the optimal level and the royalties are the primary form of payment, patent thickets may still cause royalty stacking and double marginalization. For example, license agreements for innovations in biotech industries often include both fixed fees and royalties. Nevertheless, a recent survey shows that license agreements in the biotech industry typically provide for relatively low upfront payments (a median of $50,000), and once a product is approved, the license agreement usually requires the biotechnology company to pay the research institution a percentage of its revenues from sales of that product, with a median royalty rate of 2.25% (Foley Hoag LLP, 2007). In such cases, the upfront fees can be negligible compared to the royalties, and thus royalty stacking and double marginalization may still occur.

Licensing terms and market outcomes depend on the cost structures in upstream and downstream markets. This implies that the impact of the licensing forms on market outcomes may vary for different industries. For example, in chemical and pharmaceutical industries the upstream firms incur significant marginal costs providing patented technologies to downstream firms, while information technologies, especially those related with digital products such as software and online contents, tend to have near-zero marginal costs. It will be interesting to empirically test this prediction in these industries.

There are limitations to the analysis in this article. First, the model does not include the transaction costs of negotiating license agreements. There is no doubt that negotiating separate licenses with many parties is much more costly than negotiating with one party (or a coalition of parties). The results here also imply that the lack of coordination may lead to more use of licenses that cause royalty stacking and double marginalization (namely pure quantity-based or revenue-based royalty licenses). Therefore, coordination between upstream firms can be very important in mitigating the effects of patent thickets. It must be noted, though, that even when upstream firms coordinate and negotiate licenses as one entity, choosing a proper form of license is still important, since different forms of license result in different prices and profits for firms.

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Appendix 1. One License Agreement ($M = 1$) Covering N Patents

A. Benchmark Case: No License Agreement (Fully Integrated Firm). A profit-maximizing, fully integrated firm solves the following problem:

$$\underset{p}{Max} \quad T^{II} = D(p)\left(p - a - \sum_{k=1}^{N} c^k\right).$$

We find that the optimal price is:

$$p^{II} = \frac{\varepsilon}{\varepsilon - 1}(1+m)a.$$

The maximized profit is thus given by:

$$T^{II} = \frac{A(\varepsilon-1)^{\varepsilon-1}}{[a(1+m)]^{\varepsilon-1}\varepsilon^{\varepsilon}}.$$

B. Quantity-Based Royalty Licenses. At the second stage, the downstream firm's problem is given by:

$$\underset{p}{Max}\ \Pi = D(p)(p-a-u).$$

The optimal pricing rule is: $p = \frac{\varepsilon}{\varepsilon-1}(a+u)$.

At the first stage, the integrated upstream firm solves:

$$\underset{u}{Max}\ \pi = D(p)\left(u - \sum_{k=1}^{N} c^k\right)$$

$$s.t.\ \Pi = D(p)(p-a-u) \geq \Pi^o, \text{ where } p = \frac{\varepsilon}{\varepsilon-1}(a+u).$$

We construct the Lagrangian and derive the first-order condition (FOC). From the FOC, we have: $\lambda = 1 - \frac{(u-ma)}{a+u}$, where λ is the Lagrange multiplier. When $u = \frac{1}{\varepsilon-1}(1+\varepsilon m)a$, $\lambda = 0$. By the complementary slackness condition, we know that the participation constraint is not binding. And when $u = \frac{1}{\varepsilon-1}(1+\varepsilon m)a$, $\Pi = \frac{A(\varepsilon-1)^{2\varepsilon-2}}{[a(1+m)]^{\varepsilon-1}\varepsilon^{2\varepsilon-1}}$. Define $\Pi_{up}^{IU} \equiv \frac{A(\varepsilon-1)^{2\varepsilon-2}}{[a(1+m)]^{\varepsilon-1}\varepsilon^{2\varepsilon-1}}$. This means that as long as $\Pi^o \leq \Pi_{up}^{IU}$, the constraint is not binding. Thus $u = \frac{1}{\varepsilon-1}(1+\varepsilon m)a$ is the solution when $0 \leq \Pi^o \leq \Pi_{up}^{IU}$. By plugging this solution into $p = \frac{\varepsilon}{\varepsilon-1}(a+u)$, we get the equilibrium price; and by plugging u and p into $\pi = D(p)\left(u - \sum_{k=1}^{N} c^k\right)$, we get the profit for the upstream firm for this range of Π^o.

When the participation is binding (i.e., $\Pi^o > \Pi_{up}^{IU}$), the solution is simply defined by the constraint holding with " = ": $D(p)(p-a-u) = \Pi^o$, where $p = \frac{\varepsilon}{\varepsilon-1}(a+u)$. We get: $u = (1-1/\varepsilon)\left(\frac{A}{\Pi^o\varepsilon}\right)^{1/(\varepsilon-1)} - a$. We can then get p and π. Note that $\pi = \varepsilon\Pi^o\left[1 - \left(\frac{\varepsilon\Pi^o}{A}\right)^{1/(\varepsilon-1)}(1+m)a\right] - \Pi^o$. Because the upstream firm has to earn a non-negative payoff to agree to license its patents, we must have $\pi \geq 0$. Thus we get the upper bound on Π^o by solving: $\varepsilon\Pi^o\left[1 - \left(\frac{\varepsilon\Pi^o}{A}\right)^{1/(\varepsilon-1)}(1+m)a\right] - \Pi^o \geq 0$. We get: $\Pi^o \leq \frac{A(\varepsilon-1)^{\varepsilon-1}}{[(1+m)a]^{\varepsilon-1}\varepsilon^{\varepsilon}} = T^{II}$.

Thus the solution is: $u^{IU} = \begin{cases} \frac{1+\varepsilon m}{\varepsilon-1}a, & 0 \leq \Pi^o \leq \Pi_{up}^{IU} \\ (1-1/\varepsilon)\left(\frac{A}{\Pi^o\varepsilon}\right)^{1/(\varepsilon-1)} - a, & \Pi_{up}^{IU} < \Pi^o \leq T^{II} \end{cases}$. The rest of the results can then be derived.

C. Revenue-Based Royalty Licenses. At the second stage, the downstream chooses the optimal price for any given royalty rate r by solving: $Max_p \quad \Pi = D(p)[p(1-r)-a]$. We get: $p = \frac{\varepsilon}{(\varepsilon-1)(1-r)}a$.

At the first stage, the integrated upstream firm solves:

$$\begin{aligned} \underset{r}{Max} \quad & \pi = D(p)(rp - ma) \\ s.t. \quad & \Pi = D(p)[p(1-r)-a] \geq \Pi^o, \\ & \text{where } p = \frac{\varepsilon}{(\varepsilon-1)(1-r)}a. \end{aligned}$$

Using the same method as in Appendix 1B, by the KKT conditions, when $r = \frac{m(\varepsilon-1)+1}{m(\varepsilon-1)+\varepsilon}$, the participation constraint is not binding. We can then use $r = \frac{m(\varepsilon-1)+1}{m(\varepsilon-1)+\varepsilon}$ to get the range where the constraint is not binding: when $r = \frac{m(\varepsilon-1)+1}{m(\varepsilon-1)+\varepsilon}$, $\Pi = \frac{A(\varepsilon-1)^{2\varepsilon-1}}{a^{\varepsilon-1}\varepsilon^{\varepsilon}[\varepsilon+m(\varepsilon-1)]^{\varepsilon}} \equiv \Pi_{up}^{IR}$. Thus $r = \frac{m(\varepsilon-1)+1}{m(\varepsilon-1)+\varepsilon}$ is the solution when $0 \leq \Pi^o \leq \Pi_{up}^{IR}$.

When the participation is binding (i.e., $\Pi^o > \Pi_{up}^{IR}$), the solution is defined by: $D(p)[p(1-r)-a] = \Pi^o$, where $p = \frac{\varepsilon}{(\varepsilon-1)(1-r)}a$. We get: $r = 1 - \varepsilon\left(\frac{\Pi^o}{A}\right)^{1/\varepsilon}\left(\frac{a}{\varepsilon-1}\right)^{1-1/\varepsilon}$. We can then get p and π. Note that $\pi = \Pi^o\left[\left(\frac{A}{\Pi^o}\right)^{1/\varepsilon}\left(\frac{\varepsilon-1}{a}\right)^{1-1/\varepsilon} - \varepsilon - m\varepsilon + m\right]$. We then find the upper bound for Π^o by solving: $\Pi^o\left[\left(\frac{A}{\Pi^o}\right)^{1/\varepsilon}\left(\frac{\varepsilon-1}{a}\right)^{1-1/\varepsilon} - \varepsilon - m\varepsilon + m\right] \geq 0$. We have: $\Pi^o \leq \frac{A(\varepsilon-1)^{\varepsilon-1}}{a^{\varepsilon-1}(\varepsilon+m\varepsilon-m)^{\varepsilon}} \equiv \Pi_{\max}^{IR}$.

Thus the solution is: $r^{IR} = \begin{cases} \frac{m(\varepsilon-1)+1}{m(\varepsilon-1)+\varepsilon}, & 0 \leq \Pi^o \leq \Pi_{up}^{IR} \\ 1 - \varepsilon\left(\frac{\Pi^o}{A\varepsilon}\right)^{1/\varepsilon}\left(\frac{a}{\varepsilon-1}\right)^{1-1/\varepsilon}, & \Pi_{up}^{IR} < \Pi^o \leq \Pi_{\max}^{IR} \end{cases}$. The rest of the results can then be derived.

D. Profit-Based Royalty Licenses and Fixed Fee Licenses. First, under a profit-based royalty license, the downstream firm solves: $Max_p \quad \Pi = D(p)(p-a)(1-g)$.

Thus the optimal price is given by: $p^{IG} = \frac{\varepsilon}{\varepsilon-1}a$.

The integrated upstream firm's problem is given by:

$$\begin{aligned} \underset{g}{Max} \quad & \pi = D(p)[g(p-a) - ma] \\ s.t. \quad & \Pi = D(p)(p-a)(1-g) \geq \Pi^o, \text{ where } p = \frac{\varepsilon}{\varepsilon-1}a. \end{aligned}$$

By the first-order condition, the Lagrange multiplier $\lambda = 1$. By the complementary slackness condition, the participation constraint is always binding. Thus the optimal royalty rate is determined by: $D(p)(p-a)(1-g) = \Pi^o$, where $p = \frac{\varepsilon}{\varepsilon-1}a$. We have: $g^{IG} = 1 - \frac{a^{\varepsilon-1}\varepsilon^{\varepsilon}\Pi^o}{A(\varepsilon-1)^{\varepsilon-1}}$. Plugging this optimal rate into $\pi = D(p)[g(p-a)-ma]$, we have: $\pi^{IG} = \frac{A(\varepsilon-1)^{\varepsilon-1}}{a^{\varepsilon-1}\varepsilon^{\varepsilon}}[1 - m(\varepsilon-1)] - \Pi^o$. Since the constraint is always binding, we know $\Pi^{IG} = \Pi^o$. Therefore, we have: $T^{IG} = \pi^{IG} + \Pi^{IG} = \frac{A(\varepsilon-1)^{\varepsilon-1}}{a^{\varepsilon-1}\varepsilon^{\varepsilon}}[1 - m(\varepsilon-1)]$. The upper bound on Π^o is determined by: $\pi^{IG} = T^{IG} - \Pi^o \geq 0$, that is, $\Pi^o \leq T^{IG}$. So the results here are valid for $0 \leq \Pi^o \leq T^{IG}$.

Second, under a fixed-fee license where a fixed amount F is paid by the downstream firm to the upstream firm, the downstream firm's profit is: $\Pi = D(p)(p-a) - F$. Thus the optimal price charged by the downstream firm is given by: $p^{IF} = \frac{\varepsilon}{\varepsilon-1}a$.

The integrated upstream firm solves:

$$\begin{aligned} \underset{F}{Max} \quad & \pi = F - D(p)ma \\ s.t. \quad & \Pi = D(p)(p-a) - F \geq \Pi^o, \text{ where } p = \frac{\varepsilon}{\varepsilon-1}a. \end{aligned}$$

For the same reason as in the case of profit-based royalty license, the participation constraint is always binding, $\Pi^{IF} = \Pi^o$. Thus the optimal fee is determined by: $D(p)(p-a) - F = \Pi^o$, where $p = \frac{\varepsilon}{\varepsilon-1}a$. We have:

$F^{IF} = \frac{A(\varepsilon-1)^{\varepsilon-1}}{a^{\varepsilon-1}\varepsilon^{\varepsilon}} - \Pi^o$. Plugging this optimal fee into $\pi = F - D(p)ma$, we have $\pi^{IF} = \frac{A(\varepsilon-1)^{\varepsilon-1}}{a^{\varepsilon-1}\varepsilon^{\varepsilon}}[1 - m(\varepsilon - 1)] - \Pi^o$. Thus, $T^{IF} = \pi^{IF} + \Pi^{IF} = \frac{A(\varepsilon-1)^{\varepsilon-1}}{a^{\varepsilon-1}\varepsilon^{\varepsilon}}[1 - m(\varepsilon - 1)] = T^{IG}$. And similar to the case of a profit-based royalty license, $\Pi^o \le T^{IF}$. So the results here are valid for $0 \le \Pi^o \le T^{IF}$.

E. Hybrid Licenses: Royalty plus Fixed Fee. First, consider a hybrid license with a per-unit royalty rate u and a fixed fee F. The downstream firm's payoff is given by: $\Pi = D(p)(p - a - u) - F$. Its optimal pricing decision is: $p = \frac{\varepsilon}{\varepsilon-1}(a + u)$.

The upstream firm solves the following problem:

$$\begin{aligned} \underset{u,F}{Max} \quad & \pi = D(p)(u - ma) + F \\ s.t. \quad & \Pi = D(p)(p - a - u) - F \ge \Pi^o, \\ \text{where} \quad & p = \frac{\varepsilon}{\varepsilon - 1}(a + u). \end{aligned}$$

We construct the Lagrangian and derive the first-order conditions (FOC). By the first-order condition on F, $\frac{\partial \Lambda}{\partial F} = 1 - \lambda = 0$, we know that the Lagrange multiplier $\lambda = 1$. Plug $\lambda = 1$ into $\frac{\partial \Lambda}{\partial u} = 0$, we get: $u^{IHU} = ma$. So, $p^{IHU} = \frac{\varepsilon}{\varepsilon-1}(1 + m)a$. Since $\lambda = 1$, by the complementary slackness condition, the participation constraint is always binding. Thus F is determined by: $D(p)(p - a - ma) - F = \Pi^o$, where $p = \frac{\varepsilon}{\varepsilon-1}(1 + m)a$. We have: $F^{IHU} = \frac{A(\varepsilon-1)^{\varepsilon-1}}{[a(1+m)]^{\varepsilon-1}\varepsilon^{\varepsilon}} - \Pi^o = T^{II} - \Pi^o$. Plugging $u^{IHU} = ma$, $F^{IHU} = T^{II} - \Pi^o$, and $p^{IHU} = \frac{\varepsilon}{\varepsilon-1}(1 + m)a$ into $\pi = D(p)(u - ma) + F$, we have $\pi^{IHU} = T^{II} - \Pi^o$. Since the constraint is always binding, we know $\Pi^{IHU} = \Pi^o$. And from $\pi^{IHU} = T^{II} - \Pi^o \ge 0$, we know that $\Pi^o \le T^{II}$. The valid range for Π^o is $0 \le \Pi^o \le T^{II}$.

Next, consider a hybrid license consisting of a royalty rate r for each dollar of sales revenue and a fixed fee F. Again, the downstream firm's optimal pricing rule is the same as that in a pure revenue-based royalty license: $p = \frac{\varepsilon}{(\varepsilon-1)(1-r)}a$.

The integrated upstream firm solves:

$$\begin{aligned} \underset{r,F}{Max} \quad & \pi = D(p)(rp - ma) + F \\ s.t. \quad & \Pi = D(p)[p(1 - r) - a] - F \ge \Pi^o, \\ \text{where} \quad & p = \frac{\varepsilon}{(\varepsilon - 1)(1 - r)}a. \end{aligned}$$

We construct the Lagrangian and derive the first-order conditions (FOC). Again, by the first-order condition on F, $\frac{\partial \Lambda}{\partial F} = 1 - \lambda = 0$, the Lagrange multiplier $\lambda = 1$. Plug $\lambda = 1$ into $\frac{\partial \Lambda}{\partial r} = 0$, we get: $r^{IHR} = \frac{m}{1+m}$. So, $p^{IHR} = \frac{\varepsilon}{\varepsilon-1}(1 + m)a$. Since $\lambda = 1$, by the complementary slackness condition, the participation constraint is always binding. Thus F is determined by: $D(p)\left(\frac{\varepsilon}{\varepsilon-1}a - a\right) - F = \Pi^o$, where $p = \frac{\varepsilon}{\varepsilon-1}(1 + m)a$. We have: $F^{IHR} = \frac{A(\varepsilon-1)^{\varepsilon-1}}{a^{\varepsilon-1}(1+m)^{\varepsilon}\varepsilon^{\varepsilon}} - \Pi^o = \frac{T^{II}}{1+m} - \Pi^o$. Plugging r^{IHR}, F^{IHR}, and p^{IHR} into $\pi = D(p)(rp - ma) + F$, we have $\pi^{IHR} = T^{II} - \Pi^o$. Since the constraint is always binding, we know $\Pi^{IHR} = \Pi^o$. The valid range of Π^o for this type of hybrid license is also $0 \le \Pi^o \le T^{II}$.

The solutions to the above two problems are shown in Table 4 (H in the superscript indicates a hybrid license, U indicates that the royalty part is a per-unit rate, and R indicates that the royalty rate is revenue-based).

F. M Licenses. Suppose the N patents are now owned by M firms. Suppose that the i-th upstream firm owns a set of patents denoted by I and the marginal cost of providing all these patents is $c_i = \sum_{k \in I} c^k = \sum_{k \in I} m^k a \equiv m_i a$. Under M licenses, the optimal licenses and market outcome are derived using the same approach as in Appendix 1B, 1C, and 1E, respectively. Below we list the maximization problems and the results are shown in Tables 5, 6, and 7. The details are available upon request.

When quantity-based royalty licenses are used, the downstream firm pays the i-th upstream firm a royalty of u_i (covering all the patents in I) for each unit of the final product. The downstream firm's profit is: $\Pi = D(p)(p - a - \sum_{j=1}^{M} u_j)$ and the pricing rule is: $p = \frac{\varepsilon}{\varepsilon-1}(a + \sum_{j=1}^{M} u_j)$. The i-th upstream firm solves the following problem:

$$\begin{aligned} \underset{u_i}{Max} \quad \pi_i &= D(p)(u_i - m_i a) \\ s.t. \quad \Pi &= D(p)\left(p - a - u_i - \sum_{j\neq i} u_j\right) \geq \Pi^o, \\ \text{where} \quad p &= \frac{\varepsilon}{\varepsilon - 1}\left(a + \sum_{j=1}^{M} u_j\right). \end{aligned}$$

When revenue-based royalty licenses are used, a license specifies a royalty rate of r_i to be paid to the i-th upstream firm for each dollar of revenue the downstream firm earns. The downstream firm's profit is: $\Pi = D(p)\left[p\left(1 - \sum_{i=1}^{M} r_i\right) - a\right]$. It chooses the optimal price according to $p = \frac{\varepsilon}{(\varepsilon-1)\left(1-\sum_{i=1}^{M} r_i\right)} a$. The i-th upstream firm solves the problem below and Table 6 shows the results (see Appendix 1E).

$$\begin{aligned} Max_{r_i} \quad \pi_i &= D(p)(r_i p - m_i a) \\ s.t. \quad \Pi &= D(p)\left[p\left(1 - r_i - \sum\nolimits_{j\neq i} r_j\right)\right] \geq \Pi^o, \\ \text{where} \quad p &= \frac{\varepsilon}{(\varepsilon - 1)\left(1 - \sum_{j=1}^{M} r_j\right)} a. \end{aligned}$$

When profit-based royalty licenses are used, each of the M licenses requires that the i-th upstream firm gets g_i fraction of the downstream firm's profits. The downstream firm's profit is: $\Pi = D(p)(p - a)\left(1 - \sum_{i=1}^{M} g_i\right)$. The optimal price charged by the downstream firm is given by: $p^G = \frac{\varepsilon}{\varepsilon-1} a \equiv p^{IG}$.

The profit of the i-th upstream firm is given by: $\pi_i = D(p)[g_i(p - a) - m_i a]$.

When fixed-fee licenses are used, the downstream firm pays the i-th upstream firm the amount of F_i. The downstream firm's profit is: $\Pi = D(p)(p - a) - \sum_{i=1}^{M} F_i$. The profit-maximization condition for the downstream firm is the same as that in the case of an integrated upstream firm, thus so is the optimal price: $p^F = \frac{\varepsilon}{\varepsilon-1} a \equiv p^{IF}$.

The i-th upstream firm's profit is $\pi_i = F_i - D(p)m_i a$. The licensing fees set by the N upstream firms will add up to make the downstream firm's participation constraint binding, just as in the case of one fixed-fee license. In other words, the licensing fees will not stack up.

Now we consider hybrid licenses. When each license specifies a fixed fee F_i and a per-unit royalty rate u_i, the downstream firm's payoff is given by: $\Pi = D(p)(p - a - \sum_{i=1}^{M} u_i) - \sum_{i=1}^{M} F_i$. Its pricing rule is: $p = \frac{\varepsilon}{\varepsilon-1}(a + \sum_{i=1}^{M} u_i)$.

The i-th upstream firm solves the following problem:

$$\begin{aligned} \underset{u_i, F_i}{Max} \quad \pi_i &= D(p)(u_i - m_i a) + F_i \\ s.t. \, \Pi &= D(p)\left(p - a - u_i - \sum_{j\neq i} u_j\right) - F_i - \sum_{j\neq i} F_j \geq \Pi^o, \\ \text{where}\, p &= x\frac{\varepsilon}{\varepsilon - 1}\left(a + \sum_{j=1}^{M} u_j\right). \end{aligned}$$

When each of the M licenses consists of a fixed fee F_i and a revenue-based royalty rate r_i, the i-th upstream firm's problem is:

$$\begin{aligned} \underset{r_i, F_i}{Max} \quad \pi_i &= D(p)(r_i p - m_i a) + F_i \\ s.t. \, \Pi &= D(p)\left[p\left(1 - r_i - \sum_{j\neq i} r_j\right) - a\right] - F_i - \sum_{j\neq i} F_j \geq \Pi^o, \\ \text{where}\, p &= \frac{\varepsilon}{(\varepsilon - 1)\left(1 - \sum_{j=1}^{M} r_j\right)} a. \end{aligned}$$

Table 7 shows the solutions to the above two problems.

Appendix 2. The Price of the Final Product When One License Covers N Patents

A. Prices under Different Forms of Licenses. First, we look at the price under a quantity-based royalty license. From Appendix 1A, we know that the optimal royalty rate is: $u^{IU} = \begin{cases} \frac{1+\varepsilon m}{\varepsilon-1}a, & 0 \leq \Pi^o \leq \Pi_{up}^{IU} \\ (1-1/\varepsilon)\left(\frac{A}{\Pi^o \varepsilon}\right)^{1/(\varepsilon-1)} - a, & \Pi_{up}^{IU} < \Pi^o \leq T^{II} \end{cases}$.

Since the pricing rule is: $p = \frac{\varepsilon}{\varepsilon-1}(a+u)$, we have: $p^{IU} = \begin{cases} \left(\frac{\varepsilon}{\varepsilon-1}\right)^2(1+m)a & 0 \leq \Pi^o \leq \Pi_{up}^{IU} \\ \left(\frac{A}{\Pi^o \varepsilon}\right)^{1/(\varepsilon-1)} & \Pi_{up}^{IU} < \Pi^o \leq T^{II} \end{cases}$.

Second, from Appendix 1B, we know that the optimal revenue-based royalty rate is given by: $r^{IR} = \begin{cases} \frac{m(\varepsilon-1)+1}{m(\varepsilon-1)+\varepsilon}, & 0 \leq \Pi^o \leq \Pi_{up}^{IR} \\ 1-\varepsilon\left(\frac{\Pi^o}{A\varepsilon}\right)^{1/\varepsilon}\left(\frac{a}{\varepsilon-1}\right)^{1-1/\varepsilon}, & \Pi_{up}^{IR} < \Pi^o \leq \Pi_{\max}^{IR} \end{cases}$. Since the pricing rule is: $p = \frac{\varepsilon}{(\varepsilon-1)(1-r)}a$, we have: $p^{IR} = \begin{cases} \frac{\varepsilon(\varepsilon m - m + \varepsilon)}{(\varepsilon-1)^2}a, & 0 \leq \Pi^o \leq \Pi_{up}^{IR} \\ \left(\frac{Aa}{\Pi^o(\varepsilon-1)}\right)^{1/\varepsilon}, & \Pi_{up}^{IR} < \Pi^o \leq \Pi_{\max}^{IR} \end{cases}$.

Third, from the main body of the article, we know the price under a profit-based royalty license and that under a fixed-fee license: $p^{IG} = p^{IF} = \frac{\varepsilon}{\varepsilon-1}a$ for $0 \leq \Pi^o \leq T^{IF} = T^{IG}$.

Lastly, from Appendix 1D, we know that the price under two types of hybrid licenses is the same: $p^{IHU} = p^{IHR} = \frac{\varepsilon}{\varepsilon-1}(1+m)a = p^{II}$ for $0 \leq \Pi^o \leq T^{II}$.

B. The X Axis in Figure 1. To plot these prices as functions of Π^o, we first determine where Π_{up}^{IU}, Π_{up}^{IR}, $\Pi_{\max}^{IR}$, T^{IF} ($= T^{IG}$), T^{II} lie on the Π^o axis.

First, we prove that $\Pi_{up}^{IR} < \Pi_{up}^{IU} < T^{II}$. Define $x = \frac{\Pi_{up}^{IR}}{\Pi_{up}^{IU}}$. We have: $x = \frac{\varepsilon^{\varepsilon-1}(1+m)^{\varepsilon-1}(\varepsilon-1)}{[\varepsilon+(\varepsilon-1)m]^{\varepsilon}}$. When $m=0$, $x = \frac{\varepsilon-1}{\varepsilon} < 1$. When $m>0$, $\frac{dx}{dm} < 0$. Therefore, for any $m>0$, $x < \frac{\varepsilon}{\varepsilon-1} < 1$. Thus we have $\Pi_{up}^{IR} < \Pi_{up}^{IU}$. We also have: $\frac{\Pi_{up}^{IU}}{T^{II}} = \left(\frac{\varepsilon-1}{\varepsilon}\right)^{\varepsilon-1} < 1$, so $\Pi_{up}^{IU} < T^{II}$. Therefore, $\Pi_{up}^{IR} < \Pi_{up}^{IU} < T^{II}$.

Next, we prove that $\Pi_{up}^{IR} < \Pi_{\max}^{IR} \leq T^{II}$. We have: $\frac{\Pi_{up}^{IR}}{\Pi_{\max}^{IR}} = \left(\frac{\varepsilon-1}{\varepsilon}\right)^{\varepsilon} < 1$, so $\Pi_{up}^{IR} < \Pi_{\max}^{IR}$. Then, define $y = \frac{\Pi_{\max}^{IR}}{T^{II}} = \frac{(1+m)^{\varepsilon-1}\varepsilon^{\varepsilon}}{(\varepsilon+m\varepsilon-m)^{\varepsilon}}$. When $m=0$, $y=1$ and $\frac{dy}{dm}=0$. When $m>0$, $\frac{dy}{dm} < 0$. Therefore, for any $m>0$, $y<1$. Thus we have $\Pi_{\max}^{IR} \leq T^{II}$, with " = " holding when $m=0$. Therefore, $\Pi_{up}^{IR} < \Pi_{\max}^{IR} \leq T^{II}$.

Lastly, we prove that $0 \leq T^{IF} \leq \Pi_{\max}^{IR} \leq T^{II}$. Define $z = \frac{T^{IF}}{\Pi_{\max}^{IR}}$. We have: $z = \frac{(1-m\varepsilon+m)[\varepsilon+(\varepsilon-1)m]^{\varepsilon}}{\varepsilon^{\varepsilon}}$. When $m=0$, $z=1$ and $\frac{dz}{dm}=0$. When $m>0$, $\frac{dz}{dm} < 0$. Therefore, for any $m>0$, $z<1$. Thus we have $T^{IF} \leq \Pi_{\max}^{IR}$, with " = " holding when $m=0$. Since $\Pi_{\max}^{IR} \leq T^{II}$, with " = " holding when $m=0$, we have $T^{IF} \leq \Pi_{\max}^{IR} \leq T^{II}$, with both " = " holding when $m=0$. Recall that $T^{IF} = \frac{A(\varepsilon-1)^{\varepsilon-1}}{a^{\varepsilon-1}\varepsilon^{\varepsilon}}[1-m(\varepsilon-1)]$; when $m(\varepsilon-1) > 1$, $T^{IF}=0$ (note that when $m(\varepsilon-1) > 1$, $\Pi_{up}^{IR} > 0$). Therefore, $0 \leq T^{IF} \leq \Pi_{\max}^{IR} \leq T^{II}$.

Figure 1a exhibits the case of $m>0$. In the figure, the X axis shows $\Pi_{up}^{IR} < T^{IF} < \Pi_{up}^{IU} < \Pi_{\max}^{IR} < T^{II}$. Note that T^{IF} can be anywhere on the X axis as long as $T^{IF} < \Pi_{\max}^{IR}$, and it is also possible that $\Pi_{\max}^{IR} < \Pi_{up}^{IU}$. When $m=0$, $\Pi_{up}^{IR} < \Pi_{up}^{IU} < T^{IF} = \Pi_{\max}^{IR} = T^{II}$ is always true, which is shown on the X axis in Figure 1b.

C. Comparing the Prices under Different Forms of Licenses

1) First, we discuss the case of $m=0$. When $m=0$, $p^{IHR} = p^{IHU} = p^{IF} = p^{IG} = p^{II} = \frac{\varepsilon}{\varepsilon-1}a$. When $0 \leq \Pi^o \leq \Pi_{up}^{IR}$, $p^{IU} = p^{IR} = \left(\frac{\varepsilon}{\varepsilon-1}\right)^2 a > p^{II}$. When $\Pi_{up}^{IR} \leq \Pi^o \leq \Pi_{up}^{IU}$, p^{IR} decreases with Π^o while p^{IU} remains at the level of $\left(\frac{\varepsilon}{\varepsilon-1}\right)^2 a$. When $\Pi_{up}^{IU} \leq \Pi^o \leq T^{IF} = \Pi_{\max}^{IR} = T^{II}$, both p^{IR} and p^{IU} decrease with Π^o, until $p^{IR} = p^{IU} = p^{II} = \frac{\varepsilon}{\varepsilon-1}a$ when $\Pi^o = T^{IF} = \Pi_{\max}^{IR} = T^{II}$.
2) Next, we discuss the case of $m>0$. We have $p^{IHR} = p^{IHU} = p^{II} = \frac{\varepsilon}{\varepsilon-1}(1+m)a$ and $p^{IF} = p^{IG} = \frac{\varepsilon}{\varepsilon-1}a$. Thus, $p^{IHR} = p^{IHU} = p^{II} > p^{IF} = p^{IG}$. We then prove that $p^{IU} > p^{II}$ when $0 \leq \Pi^o < T^{II}$, and $p^{IU} = p^{II}$ when $\Pi^o = T^{II}$. Clearly, when $0 \leq \Pi^o \leq \Pi_{up}^{IU}$, $p^{IU} = \left(\frac{\varepsilon}{\varepsilon-1}\right)^2(1+m)a > p^{II} = \frac{\varepsilon}{\varepsilon-1}(1+m)a$. When $\Pi_{up}^{IU} < \Pi^o \leq T^{II}$,

p^{IU} monotonically decreases with Π^o, until $p^{IU} = p^{II} = \frac{\varepsilon}{\varepsilon-1}(1+m)a$ at $\Pi^o = T^{II}$. Finally, we prove that $p^{IR} < p^{IU}$. When $0 \le \Pi^o \le \Pi_{up}^{IR}$, $p^{IU} = p_{up}^{IU} = \left(\frac{\varepsilon}{\varepsilon-1}\right)^2(1+m)a$ while $p^{IR} = p_{up}^{IR} = \frac{\varepsilon[\varepsilon(1+m)-m]}{(\varepsilon-1)^2}a$, so clearly $p_{up}^{IR} < p_{up}^{IU}$. When $\Pi_{up}^{IR} \le \Pi^o \le \Pi_{up}^{IU}$, p^{IR} decreases with Π^o while p^{IU} remains at the level of $\left(\frac{\varepsilon}{\varepsilon-1}\right)^2 a$, so obviously $p_{up}^{IR} < p_{up}^{IU}$ at $\Pi^o = \Pi_{up}^{IU}$. When $\Pi_{up}^{IU} \le \Pi^o \le \Pi_{\max}^{IR}$, define $w = \frac{p^{IR}}{p^{IU}} = \varepsilon^{1/(\varepsilon-1)}\left(\frac{a}{\varepsilon-1}\right)^{1/\varepsilon}\left(\frac{\Pi^o}{A}\right)^{1/\varepsilon(\varepsilon-1)}$. Clearly w monotonically increases with Π^o. Therefore, to prove $p_{up}^{IR} < p_{up}^{IU}$ for $\Pi_{up}^{IU} \le \Pi^o \le \Pi_{\max}^{IR}$, it suffices to prove that $w < 1$ when $\Pi^o = \Pi_{\max}^{IR}$. We have $w(\Pi^o = \Pi_{\max}^{IR}) = \left(\frac{\varepsilon}{\varepsilon+m(\varepsilon-1)}\right)^{1/(\varepsilon-1)} < 1$. So, $p_{up}^{IR} < p_{up}^{IU}$ for $\Pi_{up}^{IU} \le \Pi^o \le \Pi_{\max}^{IR}$. Furthermore, $p^{IR}(\Pi^o = \Pi_{\max}^{IR}) = \frac{\varepsilon(1+m)-m}{\varepsilon-1}a < \frac{\varepsilon(1+m)}{\varepsilon-1}a$, so $p^{IR}(\Pi^o = \Pi_{\max}^{IR}) < p^{II}$.

Appendix 3. Proof to Proposition 1, Corollary 1, and Proposition 2

A. Proof to Proposition 1.

Proposition 1: *For any reservation payoff of the downstream firm, Π^o, an integrated upstream firm's optimal licensing form is a hybrid license that consists of a fixed fee and a quantity- or revenue-based royalty rate. The optimal licenses are:*

(1) $u^{IHU} = ma = \sum_{k=1}^{N} c^k$ *and* $F^{IHU} = \frac{A(\varepsilon-1)^{\varepsilon-1}}{[a(1+m)]^{\varepsilon-1}\varepsilon^{\varepsilon}} - \Pi^o$*; or*

(2) $r^{IHR} = \frac{m}{1+m}$ *and* $F^{IHR} = \frac{A(\varepsilon-1)^{\varepsilon-1}}{a^{\varepsilon-1}(1+m)^{\varepsilon}\varepsilon^{\varepsilon}} - \Pi^o$.

Proof: To prove that a hybrid license is optimal for the integrated upstream firm, we need to prove that the upstream firm's profit under a hybrid license is higher than any other form of license. We have shown that $\pi^{IHU} = \pi^{IHR}$ and have derived the optimal hybrid licenses as in (1) and (2) of Proposition 1. So we only need to prove that: (1) $\pi^{IHU} \ge \pi^{IU}$; (2) $\pi^{IHU} \ge \pi^{IR}$; and (3) $\pi^{IHU} \ge \pi^{IF} = \pi^{IG}$.

(1) We prove that $\pi^{IHU} \ge \pi^{IU}$. Since $\pi^{IHU} = T^{II} - \Pi^o$, we just need to prove $T^{II} \ge \pi^{IU} + \Pi^o$.
Note that even though π^{IU} has two segments, it is continuous at Π_{up}^{IU}. We know that $\pi^{IU} = \pi_{up}^{IU} = const.$ when $0 \le \Pi^o \le \Pi_{up}^{IU}$. Thus, for the range of $0 \le \Pi^o \le \Pi_{up}^{IU}$, $\pi^{IU} + \Pi^o$ as a function of Π^o has a constant slope of 1 and is maximized at $\Pi^o = \Pi_{up}^{IU}$. At $\Pi^o = \Pi_{up}^{IU}$, $\pi^{IU} + \Pi^o = \pi_{up}^{IU} + \Pi_{up}^{IU}$. So we need to prove that $T^{II} \ge \pi_{up}^{IU} + \Pi_{up}^{IU}$.
When $\Pi_{up}^{IU} \le \Pi^o \le T^{II}$, we have: $\pi^{IU} + \Pi^o = \varepsilon\Pi^o\left[1 - \left(\frac{\varepsilon\Pi^o}{A}\right)^{1/(\varepsilon-1)}(1+m)a\right] \equiv T^{IU}$. Note that $T^{IU}(\Pi^o = \Pi_{up}^{IU}) = \pi_{up}^{IU} + \Pi_{up}^{IU}$. We then find that: $\left.\frac{dT^{IU}}{d\Pi^o}\right|_{\Pi^o=\Pi_{up}^{IU}} = 1$ and $\left.\frac{dT^{IU}}{d\Pi^o}\right|_{\Pi^o=T^{II}} = 0$. Furthermore, $\frac{dT^{IU}}{d\Pi^o}$ monotonically decreases with Π^o, so we know that $\frac{dT^{IU}}{d\Pi^o} > 0$ when $\Pi_{up}^{IU} < \Pi^o < T^{II}$. Therefore, T^{IU} monotonically increases with Π^o and is maximized at $\Pi^o = T^{II}$ for $\Pi_{up}^{IU} \le \Pi^o \le T^{II}$. So, clearly, $T^{IU}(\Pi^o = \Pi_{up}^{IU}) = \pi_{up}^{IU} + \Pi_{up}^{IU} < T^{IU}(\Pi^o = T^{II})$. Now all we need for proving $T^{II} \ge \pi^{IU} + \Pi^o$ is to prove that $T^{II} \ge T^{IU}(\Pi^o = T^{II})$. It turns out that $T^{IU}(\Pi^o = T^{II}) = T^{II}$.
Thus, we know that $\pi^{IHU} \ge \pi^{IU}$ with " = " holding when $\Pi^o = T^{II}$.

(2) We prove that $\pi^{IHU} \ge \pi^{IR}$. Again we just need to prove $T^{II} \ge \pi^{IR} + \Pi^o$.
Even though π^{IR} has two segments, it is continuous at Π_{up}^{IR}. We know that $\pi^{IR} = \pi_{up}^{IR} = const.$ when $0 \le \Pi^o \le \Pi_{up}^{IR}$. Thus, for $0 \le \Pi^o \le \Pi_{up}^{IR}$, $\pi^{IR} + \Pi^o$ as a function of Π^o has a constant slope of 1 and is maximized at $\Pi^o = \Pi_{up}^{IR}$. At $\Pi^o = \Pi_{up}^{IU}$, $\pi^{IR} + \Pi^o = \pi_{up}^{IR} + \Pi_{up}^{IR}$. So we need to prove that $T^{II} \ge \pi_{up}^{IR} + \Pi_{up}^{IR}$.
When $\Pi_{up}^{IR} \le \Pi^o \le \Pi_{\max}^{IR}$, we have: $\pi^{IR} + \Pi^o = A^{1/\varepsilon}\left(\frac{\varepsilon-1}{a}\right)^{1-1/\varepsilon}(\Pi^o)^{1-1/\varepsilon} - (\varepsilon-1)(1+m)\Pi^o \equiv T^{IR}$. Note that $T^{IR}(\Pi^o = \Pi_{up}^{IR}) = \pi_{up}^{IR} + \Pi_{up}^{IR}$. We then find that: $\left.\frac{dT^{IR}}{d\Pi^o}\right|_{\Pi^o=\Pi_{up}^{IR}} = 1$ and $\left.\frac{dT^{IR}}{d\Pi^o}\right|_{\Pi^o=\Pi_{\max}^{IR}} = -\frac{(\varepsilon-1)m}{\varepsilon} < 0$. Furthermore, $\frac{dT^{IR}}{d\Pi^o}$ monotonically decreases with Π^o. So we know that there exists a $\Pi^o \in (\Pi_{up}^{IR}, \Pi_{\max}^{IR})$ such that $\frac{dT^{IR}}{d\Pi^o} = 0$.

Define $\tilde{\Pi}^o$ such that $\frac{dT^{IR}}{d\Pi^o}\Big|_{\Pi^o=\tilde{\Pi}^o} = 0$. We get: $\tilde{\Pi}^o = \frac{A(\varepsilon-1)^{\varepsilon-1}}{a^{\varepsilon-1}[\varepsilon(1+m)]^{\varepsilon}}$. We know that T^{IR} is maximized at $\tilde{\Pi}^o$. Clearly, $T^{IR}(\Pi^o = \Pi^{IU}_{up}) = \pi^{IR}_{up} + \Pi^{IR}_{up} < T^{IR}(\Pi^o = \tilde{\Pi}^o)$. Now all we need for proving $T^{II} \geq \pi^{IR} + \Pi^o$ is to prove that $T^{II} \geq T^{IR}(\Pi^o = \tilde{\Pi}^o)$. It turns out that $T^{IR}(\Pi^o = \tilde{\Pi}^o) = T^{II}$.

Thus, we know that $\pi^{IHU} \geq \pi^{IR}$ with " = " holding when $\Pi^o = \tilde{\Pi}^o$.

(3) We prove that $\pi^{IHU} \geq \pi^{IF}$. Since $\pi^{IHU} = T^{II} - \Pi^o$ and $\pi^{IF} = T^{IF} - \Pi^o$, we just need to prove that $T^{II} \geq T^{IF}$. In Appendix 2B we already proved that $T^{II} \geq T^{IF}$ with " = " holding with $m = 0$.

Even though π^{IR} has two segments, it is continuous at Π^{IR}_{up}. We know that $\pi^{IR} = \pi^{IR}_{up} = const.$ when $0 \leq \Pi^o \leq \Pi^{IR}_{up}$. Thus, for $0 \leq \Pi^o \leq \Pi^{IR}_{up}$, $\pi^{IR} + \Pi^o$ as a function of Π^o has a constant slope of 1 and is maximized at $\Pi^o = \Pi^{IR}_{up}$. At $\Pi^o = \Pi^{IU}_{up}$, $\pi^{IR} + \Pi^o = \pi^{IR}_{up} + \Pi^{IR}_{up}$. So we need to prove that $T^{II} \geq \pi^{IR}_{up} + \Pi^{IR}_{up}$.

Q.E.D.

B. Proof to Corollary 1.

Corollary 1: *When $m = 0$, the optimal licensing strategy for an integrated upstream firm is a fixed-fee license with a fee of $F^{IF} = \frac{A(\varepsilon-1)^{\varepsilon-1}}{a^{\varepsilon-1}\varepsilon^{\varepsilon}} - \Pi^o$ or a profit-based royalty license with a royalty rate of $g^{IG} = 1 - \frac{a^{\varepsilon-1}\varepsilon^{\varepsilon}\Pi^o}{A(\varepsilon-1)^{\varepsilon-1}}$.*

Proof: When $m = 0$, $T^{II} = T^{IF} = T^{IG}$, so $\pi^{IHU} = \pi^{IHR} = T^{II} - \Pi^o = \pi^{IF} = \pi^{IG} = T^{IF} - \Pi^o$. The fee in the optimal fixed-fee license and the royalty rate in the optimal profit-based royalty license are given in Appendix 1C as well as in Table 3. In fact, when $m = 0$, the optimal hybrid licenses given in Proposition 1 become the optimal fixed-fee license given here. *Q.E.D.*

C. Proof to Proposition 2.

Proposition 2: *When one integrated upstream firm owns N patents and chooses the optimal licensing strategies described in Proposition 1, the equilibrium price of the downstream product is the same as that charged by a fully integrated firm: $p = \frac{\varepsilon}{\varepsilon-1}(1 + m)a$; and the sum of the profits of the downstream firm and the integrated upstream firm also equals that of a fully integrated firm.*

Proof: In Appendix 1D we have proved that the equilibrium price under both types of hybrid licenses is $p = \frac{\varepsilon}{\varepsilon-1}(1 + m)a$ and the profit of the upstream firm is $\pi^{IHU} = \pi^{IHR} = T^{II} - \Pi^o$. So $\pi^{IHU} + \Pi^o = \pi^{IHR} + \Pi^o = T^{II}$. *Q.E.D.*

Appendix 4. Equity Holding between Firms

First, suppose the integrated upstream firm owns e fraction of the equity of the downstream firm and the equity provides the upstream firm with *no other rights* than receiving e fraction of the downstream profits. The downstream firm's profit maximization problem is given by:

$$\underset{p}{Max} \quad \Pi = D(p)(p - a)(1 - e).$$

Thus the optimal price is given by $p = \frac{\varepsilon}{\varepsilon-1}a$, which is the same as that under a profit-based royalty (or a fixed-fee) license.

Next, suppose the downstream firm acquires e fraction of the integrated upstream firm's equity. The downstream firm pays a lump sum of S to the upstream firm for the equity, and gains access to upstream technologies for no further charges.

After the equity transaction, the downstream firm shares the integrated upstream firm's profit. So the upstream firm's payoff is given by:

$$\pi = S - D(p)ma(1 - e).$$

The downstream firm's profit maximization problem is:

$$\underset{p}{Max} \quad \Pi = D(p)(p - a) - eD(p)ma - S.$$

The downstream firm's optimal price is thus given by $p = \frac{\varepsilon}{\varepsilon-1}(1 + e \cdot m)a$.

Appendix 5. Proof to Propositions 3 and 4, Corollaries 2 and 3

A. Proof to Proposition 3 and Corollary 2.

Proposition 3: *When N patents are licensed through M rather than one quantity-based royalty license, royalty stacking and double marginalization occur when* $0 \leq \Pi^o < \Pi_{up}^{IU}$, *but do not occur when* $\Pi_{up}^{IU} \leq \Pi^o \leq T^{II}$.

Proof:

(1) First, we prove that $\Pi_{up}^U < \Pi_{up}^{IU}$. We have: $\Pi_{up}^U = \frac{A[(\varepsilon-1)(\varepsilon-M)]^{\varepsilon-1}}{[a(1+m)]^{\varepsilon-1}\varepsilon^{2\varepsilon-1}}$ and $\Pi_{up}^{IU} = \frac{A(\varepsilon-1)^{2\varepsilon-2}}{[a(1+m)]^{\varepsilon-1}\varepsilon^{2\varepsilon-1}}$. Clearly, $\Pi_{up}^U < \Pi_{up}^{IU}$.

(2) Next we prove that when $0 \leq \Pi^o \leq \Pi_{up}^U$, $\sum_{i=1}^M u_i^U > u^{IU}$ and $p^U > p^{IU}$. We know $\sum_{i=1}^M u_i^U = \frac{1}{\varepsilon-M}(M+m\varepsilon)a$ and $u^{IU} = \frac{1}{\varepsilon-1}(1+m\varepsilon)a \equiv u_{up}^{IU}$. It can be proved that for any $M>1$, $\sum_{i=1}^M u_i^U > u_{up}^{IU}$. We also have: $p^U = \frac{\varepsilon^2}{(\varepsilon-1)(\varepsilon-M)}(1+m)a \equiv p_{up}^U$ and $p^{IU} = \left(\frac{\varepsilon}{\varepsilon-1}\right)^2(1+m)a \equiv p_{up}^{IU}$. It's obvious that $p^U > p^{IU}$.

(3) We now prove that when $\Pi_{up}^U \leq \Pi^o < \Pi_{up}^{IU}$, $\sum_{i=1}^M u_i^U > u^{IU}$, and $p^U > p^{IU}$. In this range, $\sum_{i=1}^M u_i^U = (1-1/\varepsilon)\left(\frac{A}{\Pi^o\varepsilon}\right)^{1/(\varepsilon-1)} - a$, while $u^{IU} = u_{up}^{IU}$. It can be easily shown that $\sum_{i=1}^M u_i^U$ decreases with Π^o and $\sum_{i=1}^M u_i^U = u_{up}^{IU}$ at $\Pi^o = \Pi_{up}^{IU}$. Therefore, $\sum_{i=1}^M u_i^U > u^{IU}$ in the range $\Pi_{up}^U \leq \Pi^o < \Pi_{up}^{IU}$. Similarly, in this range $p^{IU} = p_{up}^{IU}$, while $p^U = \left(\frac{A}{\Pi^o\varepsilon}\right)^{1/(\varepsilon-1)}$, which decreases with Π^o and $p^U = p_{up}^{IU}$ at $\Pi^o = \Pi_{up}^{IU}$. Thus $p^U > p^{IU}$ for $\Pi_{up}^U \leq \Pi^o < \Pi_{up}^{IU}$.

(4) Finally, when $\Pi_{up}^{IU} \leq \Pi^o \leq T^{II}$, $\sum_{i=1}^M u_i^U = u^{IU}$, and $p^U = p^{IU}$.

Summing up (2)–(4), we see that when $0 \leq \Pi^o < \Pi_{up}^{IU}$, $\sum_{i=1}^M u_i^U > u^{IU}$ and $p^U > p^{IU}$; when $\Pi_{up}^{IU} \leq \Pi^o \leq T^{II}$, $\sum_{i=1}^M u_i^U = u^{IU}$ and $p^U = p^{IU}$.

Q.E.D.

Corollary 2: *Under M* ($1 < M \leq N$) *quantity-based royalty licenses, when royalty stacking and double marginalization do occur, the higher M is, the more severe these problems.*

Proof:

From Table 5, we know that when $0 \leq \Pi^o \leq \Pi_{up}^U$, $\sum_{i=1}^M u_i^U = \frac{1}{\varepsilon-M}(m\varepsilon+M)a$. The derivative with respect to M is $\frac{\varepsilon(1+m)}{(\varepsilon-M)^2}a > 0$. This means $\sum_{i=1}^M u_i^U$ increases with M, hence royalty stacking is more severe as M increases.

The price is given by: $p^U = \frac{\varepsilon^2}{(\varepsilon-1)(\varepsilon-M)}(1+m)a \equiv p_{up}^U$, and it is obvious that p^U increases as M increases. This means double marginalization is more severe as M increases.

We also know that Π_{up}^U decreases with M. Now suppose $M_1 < M_2$ and $\Pi_{up}^U(M_2) \leq \Pi^o \leq \Pi_{up}^U(M_1)$. In this range, it can be shown that $\sum_{i=1}^{M_1} u_i^U < \sum_{i=1}^{M_2} u_i^U$ and $p^U(M_1) < p^U(M_2)$ (the logic is similar to Step 3 in the proof to Proposition 3).

Q.E.D.

B. Proof to Proposition 4 and Corollary 3.

Proposition 4: *When N patents are licensed through N rather than one revenue-based royalty licenses, royalty stacking and double marginalization occur when* $0 \leq \Pi^o < \Pi_{up}^{IR}$, *but do not occur when* $\Pi_{up}^{IR} \leq \Pi^o \leq \Pi_{\max}^{IR}$.

Proof:

(1) First, we prove that $\Pi_{up}^R < \Pi_{up}^{IR}$. We have: $\Pi_{up}^R = \frac{A(\varepsilon-1)^{2\varepsilon-1}}{a^{\varepsilon-1}\varepsilon^\varepsilon[(1+m)(\varepsilon-1)+M]^\varepsilon}$ and $\Pi_{up}^{IR} = \frac{A(\varepsilon-1)^{2\varepsilon-1}}{a^{\varepsilon-1}\varepsilon^\varepsilon[\varepsilon+m(\varepsilon-1)]^\varepsilon}$. Clearly, $\Pi_{up}^R < \Pi_{up}^{IR}$.

(2) Next we prove that when $0 \leq \Pi^o \leq \Pi_{up}^R$, $\sum_{i=1}^M r_i^R > r^{IR}$ and $p^R > p^{IR}$. We know $\sum_{i=1}^M r_i^R = \frac{m(\varepsilon-1)+M}{(1+m)(\varepsilon-1)+M}$ and $r^{IR} = \frac{m(\varepsilon-1)+1}{m(\varepsilon-1)+\varepsilon} \equiv r_{up}^{IR}$. It can be proved that for any $M>1$, $\sum_{i=1}^M r_i^R > r_{up}^{IR}$. We also have: $p^R = \frac{\varepsilon}{(\varepsilon-1)^2}[(1+m)(\varepsilon-1)+M]a \equiv p_{up}^R$ and $p^{IR} = \frac{\varepsilon(\varepsilon m-m+\varepsilon)}{(\varepsilon-1)^2}a \equiv p_{up}^{IR}$. We can see that $p^R > p^{IR}$.

(3) We now prove that when $\Pi_{up}^R \leq \Pi^o < \Pi_{up}^{IR}$, $\sum_{i=1}^M r_i^R > r^{IR}$, and $p^R > p^{IR}$. In this range, $r^{IR} = r_{up}^{IR}$, while $\sum_{i=1}^M r_i^R = 1 - \varepsilon\left(\frac{\Pi^o}{A}\right)^{1/\varepsilon}\left(\frac{a}{\varepsilon-1}\right)^{1-1/\varepsilon}$ decreases with Π^o and $\sum_{i=1}^M r_i^R = r_{up}^{IR}$ at $\Pi^o = \Pi_{up}^{IR}$. Therefore,

$\sum_{i=1}^{M} r_i^R > r^{IR}$ in the range $\Pi_{up}^R \le \Pi^o < \Pi_{up}^{IR}$. Similarly, in this range, p^{IR} remains constant with $p^{IR} = p_{up}^{IR}$, while $p^R = \left[\frac{Aa}{\Pi^o(\varepsilon-1)}\right]^{1/\varepsilon}$ decreases with Π^o and $p^R = p_{up}^{IR}$ at $\Pi^o = \Pi_{up}^{IR}$. Thus $p^R > p^{IR}$ for $\Pi_{up}^R \le \Pi^o < \Pi_{up}^{IR}$.

(4) Finally, when $\Pi_{up}^{IR} \le \Pi^o \le \Pi_{\max}^{IR}$, $\sum_{i=1}^{M} r_i^R = r^{IR}$ and $p^R = p^{IR}$.

Summing up (2)–(4), we see that when $0 \le \Pi^o < \Pi_{up}^{IR}$, $\sum_{i=1}^{M} r_i^R > r^{IR}$ and $p^R > p^{IR}$; when $\Pi_{up}^{IR} \le \Pi^o \le \Pi_{\max}^{IR}$, $\sum_{i=1}^{M} r_i^R = r^{IR}$ and $p^R = p^{IR}$.

Q.E.D.

Corollary 3: *Under M ($1 < M \le N$) revenue-based royalty licenses, when royalty stacking and double marginalization do occur, the higher M is, the more severe these problems.*

Proof:

From Table 7, we know that when $0 \le \Pi^o \le \Pi_{up}^R$, $\sum_{i=1}^{M} r_i^R = \frac{m(\varepsilon-1)+M}{(1+m)(\varepsilon-1)+M}$. The derivative with respect to M is $\frac{\varepsilon-1}{[(1+m)(\varepsilon-1)+M]^2} > 0$. This means $\sum_{i=1}^{M} r_i^R$ increases with M, in other words, royalty stacking is more severe as M increases.

The price is given by: $p^R = \frac{\varepsilon}{(\varepsilon-1)^2}[(1+m)(\varepsilon-1)+M]a \equiv p_{up}^R$, and it is obvious that p^R increases as M increases. This means double marginalization is more severe as M increases.

We also know that Π_{up}^R decreases with M. Now suppose $M_1 < M_2$ and $\Pi_{up}^R(M_2) \le \Pi^o \le \Pi_{up}^R(M_1)$. In this range, it can be shown that $\sum_{i=1}^{M_1} r_i^R < \sum_{i=1}^{M_2} r_i^R$ and $p^R(M_1) < p^R(M_2)$ (the logic is similar to Step 3 in the proof to Proposition 4).

Q.E.D.

Appendix 6. Proof to Proposition 8 and Corollary 4

A. Proof to Proposition 8.

Proposition 8: *The optimal hybrid licenses yield the highest total profits of the upstream and downstream firms, which are equal to the total profit of a fully integrated firm.*

Proof: From Table 7, we know that $\sum_{i=1}^{M} \pi_i^{HU} + \Pi^{HU} = \sum_{i=1}^{M} \pi_i^{HR} + \Pi^{HR} = T^{II}$. In Appendix 3, we have already proved that $T^{II} \ge \pi^{IU} + \Pi^{IU}$, $T^{II} \ge \pi^{IR} + \Pi^{IR}$, and $T^{II} \ge T^{IF} = \pi^{IF} + \Pi^{IF} = \pi^{IG} + \Pi^{IG}$. For profit-based royalty and fixed-fee licenses, we also have: $\sum_{i=1}^{M} \pi_i^G + \Pi^G = \sum_{i=1}^{M} \pi_i^F + \Pi^F = T^{IF}$. So we just need to prove that $T^{II} \ge \sum_{i=1}^{M} \pi_i^U + \Pi^U$ and $T^{II} \ge \sum_{i=1}^{M} \pi_i^R + \Pi^R$.

(1) We first show that $T^{II} \ge \sum_{i=1}^{M} \pi_i^U + \Pi^U$. When $0 \le \Pi^o \le \Pi_{up}^U$, we can prove that $\pi_{up}^{IU} > \sum_{i=1}^{M} \pi_{i,up}^U$ and $\Pi_{up}^{IU} > \Pi_{up}^U$. In Appendix 3A we have proved that $T^{II} > \pi_{up}^{IU} + \Pi_{up}^{IU}$, so, $T^{II} > \sum_{i=1}^{M} \pi_{i,up}^U + \Pi_{up}^U$. When $\Pi_{up}^U \le \Pi^o \le T^{II}$, we have: $\sum_{i=1}^{M} \pi_i^U + \Pi^U = \varepsilon\Pi^o\left[1 - \left(\frac{\varepsilon\Pi^o}{A}\right)^{1/(\varepsilon-1)}(1+m)a\right] \equiv T^U$. We then find that: $\frac{dT^U}{d\Pi^o}$ monotonically decreases with Π^o, and $\left.\frac{dT^U}{d\Pi^o}\right|_{\Pi^o=\Pi_{up}^U} = N$ and $\left.\frac{dT^{IU}}{d\Pi^o}\right|_{\Pi^o=T^{II}} = 0$. So we know that $\frac{dT^U}{d\Pi^o} > 0$ when $\Pi_{up}^U < \Pi^o < T^{II}$. Therefore, T^U is maximized at $\Pi^o = T^{II}$ for $\Pi_{up}^U \le \Pi^o \le T^{II}$. We have: $T^U(\Pi^o = T^{II}) = T^{II}$. So, $T^{II} \ge \sum_{i=1}^{M} \pi_i^U + \Pi^U$ with "=" holding at $\Pi^o = T^{II}$.

(2) Next, we show that $T^{II} \ge \sum_{i=1}^{M} \pi_i^R + \Pi^R$. When $0 \le \Pi^o \le \Pi_{up}^R$, we can prove that $\pi_{up}^{IR} > \sum_{i=1}^{M} \pi_{i,up}^R$ and $\Pi_{up}^{IR} > \Pi_{up}^R$. In Appendix 3A we have proved that $T^{II} > \pi_{up}^{IR} + \Pi_{up}^{IR}$, so, $T^{II} > \sum_{i=1}^{M} \pi_{i,up}^R + \Pi_{up}^R$. When $\Pi_{up}^R \le \Pi^o \le \Pi_{\max}^{IR}$, we have: $\sum_{i=1}^{M} \pi_i^R + \Pi^R = A^{1/\varepsilon}\left(\frac{\varepsilon-1}{a}\right)^{1-1/\varepsilon}(\Pi^o)^{1-1/\varepsilon} - (\varepsilon-1)(1+m)\Pi^o \equiv T^R$. We find that: $\frac{dT^R}{d\Pi^o}$ monotonically decreases with Π^o, with $\left.\frac{dT^{IR}}{d\Pi^o}\right|_{\Pi^o=\Pi_{up}^R} = N$ and $\left.\frac{dT^R}{d\Pi^o}\right|_{\Pi^o=\Pi_{\max}^{IR}} = -\frac{(\varepsilon-1)m}{\varepsilon} < 0$. So we know that there exists a $\Pi^o \in \left(\Pi_{up}^R, \Pi_{\max}^{IR}\right)$ such that $\frac{dT^R}{d\Pi^o} = 0$. We find that T^R, just like T^{IR}, is maximized at $\tilde{\Pi}^o$ where

$\tilde{\Pi}^o = \frac{A(\varepsilon-1)^{\varepsilon-1}}{a^{\varepsilon-1}[\varepsilon(1+m)]^{\varepsilon}}$. We know that $T^{II} = T^R(\Pi^o = \tilde{\Pi}^o)$. So, $T^{II} \geq T^R = \sum_{i=1}^{M} \pi_i^R + \Pi^R$ with " = " holding when $\Pi^o = \tilde{\Pi}^o$.

Q.E.D.

B. Proof to Corollary 4.

Corollary 4: *By switching from M quantity-based royalty licenses (or M revenue-based royalty licenses) to M hybrid licenses, each firm may achieve the same or higher profit.*

Proof: First, consider the case of switching from M quantity-based royalty licenses to M hybrid licenses. In the proof to Proposition 8, we have shown that $T^{II} \geq \sum_{i=1}^{M} \pi_i^U + \Pi^U$. The firms can switch to M hybrid licenses such that the downstream firm and the first $M-1$ upstream firm earn the same profit as before, while the M-th upstream firm earns $T^{II} - \sum_{i=1}^{M-1} \pi_i^U - \Pi^U$, which is either the same as or higher than its profit before. Such hybrid licenses can be constructed by setting the royalty rates and fixed fees according to Table 7. Similarly, since we know that $T^{II} \geq \sum_{i=1}^{M} \pi_i^R + \Pi^R$, we can construct M hybrid license to replace M revenue-based royalty licenses and let each firm earn the same or higher profit. Q.E.D.